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INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATIONS

Existence and Construction of Generalized Solutions of Nonlinear Volterra Integral Equations of the First Kind

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INTRODUCTION

Consider the nonlinear integral equation

$$\int_0^t K(t,s)\left(x(s)+g\left(s^l x(s),s\right)\right)ds = f(t), \tag{1}$$

where the kernel K, as well as g and f, is an analytic function in a neighborhood of zero; moreover,

$$K(t,s) = \sum_{i=0}^{n} K_{n-i}^{n} t^{n-i} s^i + O\left((|t|+|s|)^{n+1}\right), \quad K_{n-i}^{n} \neq 0, \quad i=0,1,\ldots,n, \quad n<l.$$

We construct generalized solutions with singular part supported at a point [1] in the form

$$x(t) = c_0\delta(t) + c_1\delta^{(1)}(t) + \cdots + c_n\delta^{(n)}(t) + u(t), \tag{2}$$

where $\delta(t)$ is the Dirac function and $u(t)$ is a regular function. Generalized solutions of linear Volterra integral equations of the first kind were considered in [2–6]. A number of main results of these papers follow from the present paper. Unlike classical solutions (see the bibliography in [8–10] etc.), generalized solutions of nonlinear Volterra equations have not been studied yet.

DETERMINATION OF THE SINGULAR COMPONENT

The set of all infinitely differentiable compactly supported functions with supports in the neighborhood $(-\varrho,\varrho)$ is denoted by $D_{(-\varrho,\varrho)}$. The set of linear continuous functionals defined on $D_{(-\varrho,\varrho)}$ is denoted by $D'_{(-\varrho,\varrho)}$, and the subset of elements of the form (2) with an nth-order singularity supported at zero is denoted by $D'_{n(-\varrho,\varrho)}$. Thus the solution (2) of Eq. (1) is sought in the class $D'_{n(-\varrho,\varrho)}$ and should satisfy Eq. (1) in the sense of Sobolev–Schwartz distributions [1, pp. 49–51]. Note that the product $t^l x = t^l u(t)$ is a regular function for $n<l$ and for all $x \in D'_{n(-\varrho,\varrho)}$, which provides a solution of the problem of nonlinear operations with such distributions for Eq. (1) with $l>n$.

Since the identities

$$t^{k-i}\Theta * s^i\delta^{(j)}(s) = (-1)^j j! t^{k-j}\delta_{ij}$$

are valid in the space D' for $i,j=0,1,\ldots,n$, $k\geq n$, where Θ is the Heaviside function and δ_{ij} is the Kronecker delta, we have the relation

$$\int_0^t \sum_{k=n}^{\infty}\sum_{i=0}^{k} K_{k-i}^{k} t^{k-i} s^i \left(c_0\delta(s)+\cdots+c_n\delta^{(n)}(s)\right)ds = \sum_{j=0}^{n}(-1)^j j! \sum_{k=n}^{\infty} K_{k-j}^{k} t^{k-j} c_j.$$

Note that

$$\sum_{j=0}^{n}(-1)^j \left.\frac{\partial^j K(t,s)}{\partial s^j}\right|_{s=0} = \sum_{j=0}^{n}(-1)^j j! \sum_{k=n}^{\infty} K_{k-j}^{k} t^{k-j}.$$

Therefore, an element $x \in D'_{n(-\varrho,\varrho)}$ can be a solution of Eq. (1) only if the regular component in the representation (2) satisfies the equation

$$\int\limits_0^t K(t,s)\left(u(s)+g\left(s^l u(s),s\right)\right) ds = r\left(t,c_0,\ldots,c_n\right), \tag{3}$$

where

$$r\left(t,c_0,\ldots,c_n\right) = f(t) - \sum_{j=0}^{n}(-1)^j \frac{\partial^j K(t,0)}{\partial s^j} c_j.$$

We find the constants $c_0,\ldots,c_n$ from the system of the linear algebraic equations

$$r_t^{(i)}\left(0,c_0,\ldots,c_n\right)=0, \qquad i=0,\ldots,n, \tag{4}$$

with lower triangular matrix with diagonal entries $K_0^n, K_1^n, \ldots, K_n^n$.

If these numbers are nonzero, then the constants $c_n,\ldots,c_0$ can be found successively and uniquely. If some of the diagonal entries are zero and the vector $\left\{f(0), f'(0), \ldots, f^{(n)}(0)\right\}'$ satisfies the solvability condition, then part of the constants on the right-hand side in Eq. (3) can remain arbitrary.

Remark. If $f^{(i)}(0)=0$, $K_i^n=0$, $i=0,1,\ldots,k-1$, and $K_i^n \neq 0$ for $i=k,\ldots,n$, then, by setting $c_k=\cdots=c_n=0$, one can uniquely find the constants $c_0,\ldots,c_{k-1}$ from system (4).

DETERMINATION OF THE REGULAR COMPONENT

To construct the regular function $u(t)$, we solve Eq. (3) for the found values of $c_0,\ldots,c_n$ by combining the method of indeterminate coefficients with the successive approximation method.

For brevity, we introduce the notation

$$\Phi(u,t) := \int\limits_0^t K(t,s)\left(u(s)+g\left(s^l u(s),s\right)\right) ds - r(t,c) = 0. \tag{5}$$

Suppose that the homogeneous equation

$$\int\limits_0^t \sum_{i=0}^{n} K_{n-i}^n t^{n-i} s^i x(s) ds = 0 \tag{6}$$

has only the trivial solution. This is the case if $\sum_{i=0}^{n} K_{n-i}^n (i+j)^{-1} \neq 0$ for $j=1,2,\ldots$ Then for each positive integer N, there exist constants u_i such that

$$\left|\Phi\left(u_0+u_1 t+\cdots+u_N t^N, t\right)\right| = O\left(|t|^{n+N+1}\right). \tag{7}$$

Since the homogeneous equation (6) has only the trivial solution, we have

$$\int\limits_0^t \sum_{i=0}^{n} K_{n-i}^n t^{n-i} s^{i+j} ds \neq 0$$

for $j = 0, 1, 2, \ldots$, and the coefficients u_i are uniquely determined by the method of indeterminate coefficients after the substitution of the polynomial

$$u^0(t) = u_0 + u_1 t + \cdots + u_N t^N$$

into Eq. (5).

Next, we substitute the function

$$u(t) = u^0(t) + t^N v(t) \tag{8}$$

into Eq. (5) and collect terms containing the powers t^i, $i = n, n+1, \cdots, n+N$, taking into account relation (4) and the definition of the polynomial $u^0(t)$. Then we differentiate the resulting relation with respect to t and find the function v by the successive approximation method from the integral equation

$$v = F(v, t). \tag{9}$$

Here

$$F(v,t) = \frac{1}{K(t,t)t^N} \Bigg\{ -K(t,t)\left(u^0(t) + g\left(t^l u^0(t) + t^{l+N} v(t), t\right)\right) - \int_0^t K_t'(t,s)\left(u^0(s) + s^N v(s) + g\left(s^l u^0(s) + s^{l+N} v(s), s\right)\right) ds + r_t'(t,c) \Bigg\}.$$

Suppose that

$$\sum_{i=0}^{n} K_{n-i}^n = a \neq 0.$$

Let us show that the operator F satisfies the assumptions of the contraction mapping principle in the ball $\|x\| \le r$ of the space $C_{[-\varrho,\varrho]}$ for sufficiently large N. Indeed,

$$\left| g\left(s^l\left(u^0(s) + s^N v_1(s)\right), s\right) - g\left(s^l\left(u^0(s) + s^N v_2(s)\right), s\right) \right| \le |s|^{l+N} C_1 \left| v_1 - v_2 \right|$$

for all v_1 and v_2 in the ball $S(0,r) \subset C_{[-\varrho,\varrho]}$.

Further, since

$$|K_t'(t,s)| \le C_2(|t| + |s|)^{n-1},$$

we have

$$\left| \frac{1}{t^{n+N}} \int_0^t K_t'(t,s) s^N ds \right| \le \frac{2^{n-1} C_2}{N+1}.$$

By virtue of these estimates, there exists a constant c such that

$$\left| F(v_1, t) - F(v_2, t) \right| \le \frac{c}{N+1} \left\| v_1 - v_2 \right\|.$$

We fix a $q < 1$ and take $N > c/q - 1$. Then F is a contraction operator with exponent q in the ball $\|v\| \le r$ of the space $C_{[-\varrho,\varrho]}$. Since, by (7), $|F(0,t)| = O(|t|)$, it follows that there exists a $\tilde{\varrho} \in (0, \varrho]$ such that $\max_{|t| \le \tilde{\varrho}} |F(0,t)| \le (1-q)r$. Consequently, the contraction operator F maps the ball $\|v\| \le r$ of the space $C_{[-\tilde{\varrho},\tilde{\varrho}]}$ into itself. This implies the following assertion.

Theorem 1. *Let $l > n$, and let*

$$\sum_{i=0}^{n} K_{n-i}^n \frac{1}{i+j} \neq 0, \qquad j = 1, 2, \ldots, \qquad K_{n-i}^n \neq 0, \quad i = 0, 1, \ldots, n, \qquad \sum_{i=0}^{n} K_{n-i}^n \neq 0.$$

Then Eq. (1) *has a unique solution* (2), (8) *in the class $D_n'(-\tilde{\varrho}, \tilde{\varrho})$, where the constants $c_0, \ldots, c_n$ are found from Eq.* (4), *the coefficients $u_0, \ldots, u_N$ are computed by the method of indeterminate coefficients from Eq.* (5), *and the continuous function $v(t)$ is constructed by the successive approximation method from Eq.* (9).

Remark 1. In the analytic case, the entire regular part $u(t)$ of the solution (2) is an analytic function in a neighborhood of zero, and its Taylor coefficients can be found by the method of indeterminate coefficients from Eq. (5).

Remark 2. Instead of the analyticity of K, g, and f, in Theorem 1 one could only require that these functions are sufficiently smooth.

Remark 3. If $f^{(i)}(0) = 0$, $i = 0, 1, \ldots, n$, then all $c_i = 0$ in the solution (2), and this solution is classical.

Remark 4. If, under the assumptions of Theorem 1, some of the elements K_i^n, $i = 0, \ldots, n$, are zero and, in addition, system (4) is solvable, then the solution (2), (8) depends on k arbitrary constants, where $k = n + 1 - r$ and r is the rank of the matrix of the linear system (4).

Theorem 1 can be strengthened as follows.

Theorem 2. *Under the assumptions of Theorem* 1, *let* $\sum_{i=0}^{n} K_{n-i}^n = 0$ *and moreover, let*

$$\left.\frac{\partial^i K(t,s)}{\partial t^i}\right|_{s=t} = 0, \quad i = 0, 1, \ldots, p-1, \qquad \left.\frac{\partial^p K(t,s)}{\partial t^p}\right|_{s=t} = O\left(t^{n-p}\right), \quad p \le n.$$

Then the assertion of Theorem 1 *remains valid.*

The proof of Theorem 2 can be performed in a similar way. We only note that, by the Taylor formula, $K(t,s) = (t-s)^p Q(t,s)$ under the assumptions of Theorem 2, where $|Q(t,t)| = O\left(|t|^{n-p}\right)$. Therefore, for the construction of the equation for the function v, one should differentiate Eq. (3) $p+1$ times and set

$$F(v,t) = \frac{1}{K_t^{(p)}(t,t)t^N}\left\{ -K^{(p)}(t,t)\left(u^0(t) + g\left(t^l u^0(t) + t^{l+N} v(t), t\right)\right) \right.$$
$$\left. - \int_0^t K_t^{(p+1)}(t,s)\left(u^0(s) + s^N v(s) + g\left(s^l u^0(s) + s^{l+N} v(s), s\right)\right) ds + r_t^{(p+1)}(t,c) \right\}$$

in the corresponding equation of the form (9).

Example. Consider the equation

$$\int_0^t \left(t^2 + ts - 2s^2\right)\left(x(s) + s^5 x^2(s)\right) ds = 1 + t + t^2 + t^3.$$

Here the assumptions of Theorem 2 are valid for $n = 2$, $l = 5/2$, and $p = 1$. In the class D', we have the solution

$$x(t) = \delta(t) - \delta^{(1)}(t) - \frac{1}{4}\delta^{(2)}(t) + \frac{-1 + \sqrt{1 + t^5 \times 24/5}}{2t^5},$$

whose singular part is determined by (4) and whose regular part has been found from the equation

$$\int_0^t \left(t^2 + ts - 2s^2\right)\left(u(s) + s^5 u^2(s)\right) ds = t^3. \tag{10}$$

Equation (10) has the analytic solution

$$u(t) = \frac{-1 + \sqrt{1 + t^5 \times 24/5}}{2t^5}.$$

By Remark 1, the Taylor coefficients of the solution at the point $t = 0$ can be computed by the method of indeterminate coefficients. Note that, along with this solution, Eq. (10) has the solution

$$u_2(t) = \frac{-1 - \sqrt{1 + t^5 \times 24/5}}{2t^5}$$

for which the point $t = 0$ is a fifth-order pole.

In the general case, Eq. (1) can have several branching solutions. Such solutions can be constructed with the use of the results of the present paper in combination with well-known methods of branching theory [13, pp. 34–60].

Remark. By virtue of the results in [4, 11] on distributions in Banach spaces, Theorems 1 and 2 can be generalized to systems and integro-operator equations of the form (1) in which K is a linear kernel and g is a nonlinear mapping of Banach spaces. These results can be used in the development of the theory as well as in applications of operator-differential equations with Fredholm operator in the leading part [11, 12] to problems of nonlinear dynamics and identification [7, 8] and other problems that can be reduced to Volterra integral equations of the first kind.

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