APPLYING THE VARIATIONAL MAXIMUM PRINCIPLE TO A MODEL OF AGE-SPECIFIC DRUG INITIATION

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Consider the following problem
$\frac{\partial P(t,x)}{\partial t} + \frac{\partial P(t,x)}{\partial x} = -\mu(t,x,w(t,x))P(t,x),$

This is a model of public health intervention that seeks to prevent initiation of some dangerous behavior (e.g. drug use here) with minimum expenditures.

By $P(t,x)$ we denote the number of non-drug-users ages x years at time t. For public health applications this is the population to be protected from initiation. We assume that age-distribution of non-users at time zero is given and that cohort size effects can be ignored (see eq.(2)). For the sake of simplicity we ignor the mortality rate and migration, than the dinamics of the number of non-users can be described by the partial differential equation (1). The decision variable is $w(t,x)$, the amount of effort to expend on interventions at time t directed at individuals who are x year old. The initiations rate $\mu(t,x,w)$ is a function of this level of effort. Cost functional J measures total expenditures and demage of the initiation, r is a dicount factor, ω is a maximum age, k indicates the monetary demage of one initiation per t, $T > \omega$ can be arbitrary large. Prof. Dr. G. Feichtinger set and described the model (1)-(4).

This is a simple model and function μ can be of different form. For instance, in the case of educational measures in school classes with respect to light drugs and if we assume that preventions programs suppress drug use only at time they are run, we have the following possible form of μ:

where the rates (by age) of initiating a drug use: $\mu 0(x)$ - in the absence of a prevention program, $\nu(x)$ - in the presence of infinity intensive intervention to reduce initiation; ϵ is the efficiency of control w.

Equation (1) is a half-linear hyperbolic one and it has characteristics. Then the variational maximum principle (V.M.P.) from [1] is necessary optimality condition. V.M.P. is one of the first order necessary conditions, strengthening traditional optimality condition of the type of maximum principle. V.M.P. corresponds to deeper minimume type than finite-dimensional one (and the latter is the consequense of V.M.P.). V.M.P. also gives sufficient condition for our simple problem.

By $\eta(t,x) = (\tau + t - x, \tau)$ let us denote the characteristic of equation (1), passing through the point $(t,x) \in \Pi$ and devide Π into 3 domains:

Let the entry point of characteristic 1111 w.r.t. Π be 11111 and the exit point be 1111, where $i(t,x) = x - t$ if $(t,x) \in \Pi_1$, $i(t,x) = 0$ if $(t,x) \in \Pi_2 \cup \Pi_3$ and $e(t,x) = \omega$ if $(t,x) \in \Pi_1 \cup \Pi_2$, $e(t,x) = T - t + x$ if $(t,x) \in \Pi_3$.

Due to V.M.P., if the pair $(P^\star(t,x), w^\star(t,x))$ is optimal in problem (1)-(4) then for a.e. $(t,x) \in \Pi$ pair $(z^\star_{(t,x)} = \P^\star(\eta_{(t,x)}(\tau)), v^\star_{(t,x)}(\tau) = w^\star(\eta_{(t,x)}(\tau)))$, is optimal in the following optimal control problem 111 1111 1111 1111

If we have found optimal solution for problem (5)-(8) $(z^\star_{(t,x)}(\tau), v^\star_{(t,x)}(\tau))$ for $(t,x) \in \Pi$ then $(P^\star(t,x) = z^\star_{(t,x)}(x), w^\star(t,x) = v^\star_{(t,x)}(x))$ is the optimal solution of the initial problem (1)-(4).

Note that in equation (5) function $\mu(\eta_{(t,x)}(\tau), v(\tau) = (\mu_0(\tau) - \nu(\tau))e^{-\epsilon v(\tau)} + \nu(\tau)$ and it can depend on (t,x) by $v(\tau)$ only (as in solution of problem (5)-(8)).

Note that our problem (1)-(4) has only one half-linear equation and it may look as if V.M.P. is trivial reduction to the ordinary problem along the characteristics. However, V.M.P. works when pointwise M.P. is not effective or degenerated or unapplicable (as in case of the one-parameter control, that we find in Π_2 domain in our problem).

Consider separately the solutions of (5)-(8) in domains Π_1, Π_2 and Π_3.

If $(t,x) \in \Pi_1$ than $\eta_{(t,x)}(\tau) = \eta_{(0,x-t)}(\tau)$ and we can see that $v^\star_{(t,x)}(\tau) = v^\star_{(0,x-t)}(\tau)$ and $z^\star_{(t,x)}(\tau) = z^\star_{(0,x-t)}(\tau)$. Thus $w^\star(\tau - x + t, \tau) = v^\star_{(0,x-t)}(\tau)$ for $x - t \le \tau \le \omega$. Let $\tau = x$, then $w^\star(t,x) = v^\star_{(0,x-t)}(x)$ for $(t,x) \in \Pi_1$ or $w^\star(x-y,x) = v^\star_{(0,y)}(x)$ for $0 \le y \le \omega, y \le x \le \omega$. This form is more convinient for numerical calcucations. Similarly, $P^\star(t,x) = z^\star_{(0,x-t)}(x)$ for $(t,x) \in \Pi_1$ or $P^\star(x-y,x) = z^\star_{(0,y)}(x)$.