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M. V. Falaleev, N. A. Sidorov, and D. N. Sidorov

GENERALIZED SOLUTIONS OF VOLTERRA INTEGRAL EQUATIONS OF THE FIRST KIND

(submitted by A. M. Elizarov)

Abstract. In this paper we derived the explicit structure of generalized solutions of the Volterra integral equations of the first kind. The solution contains singular and regular components. These components can be constructed separately. On the first stage we construct the singular component of the solution by solving the special linear algebraic system. On the second stage the regular component of generalized solution can be constructed.

1. Introduction

A number of important engineering problems in electrical engineering [3], in modeling of dynamic impulse systems [11], and in identification of nonlinear dynamic systems [6], [2] can be represented as solution of the Volterra integral equations of the first kind which does not have classical continuous solutions. In some cases solutions of the algebra-differential equations and differential-operator equations with irreversible operator in the main part can be also represented via generalized solution of the

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Volterra integral equations of the first kind. The reader may see chapter 5 in the monograph [8] for details.

It is to be noted that solutions in the classes of generalized functions have clear physical sense [10], [3]. Consequently, the problems of existence, deriving and numeric computing of generalized solutions of the Volterra integral equations of the first kind are crucial in a number of important problems appear in applied mathematics.

We omit the computational part in our paper and concentrate on the structure of the generalized solutions to outline the main steps of the algorithm.

Let us consider the Volterra integral equation of the first kind

$$\int_0^t K(t,s)x(s)ds = f(t), t \geq 0, \tag{1}$$

where $K(t,s)$ and $f(t)$ are infinitely differentiable functions. If $f(0) \neq 0$, then (1) does not have classic solutions and it is reasonable to look for the solution in the distribution space [10]. Distribution space provides existence of solution and follows the physical sense of the problem [10]. For example, we can use the special combination of Dirac functions with deviating arguments as test signals for identification of nonlinear dynamical systems [6], [2], [1]. In this case it is useful to construct generalized solutions of the Volterra equations [1]. Generalized solution is the basis of mathematical models formulated in terms of impulses theory [11]. Various well-known electrical engineering problems [3] can be formulated in terms of such theory.

Generalized solutions of the Volterra integral equations of the first kind were considered in papers [5], [9], [7]. In paper [4] and in monograph [8] the generalized solutions of the singular differential-operator equations are considered. In this case such equations are reducible to the Volterra integral equations of the first kind.

In this paper we continue these studies and generalize our results [9], [7].

2. Problem Statement in the Class of Generalized Functions

For any function $K(\tau,t) \in C^\infty(R^2)$ and for any generalized function $x(t) \in D'_+$ [10] we define new generalized function $[\Theta(t) * K(\tau,t)x(t)]_{\tau=t}$. This function operates on the base of the function $\phi(t) \in D(R^1)$ and follows the rule:

$$\big((\Theta(t) * K(\tau,t)x(t))|_{\tau=t}, \phi(t)\big) = \left(x(t), \int_t^\infty K(\tau,t)\phi(\tau)d\tau\right), \qquad (2)$$

where

$$\Theta(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0. \end{cases}$$

Function

$$\psi(t) = \int_t^\infty K(\tau,t)\phi(\tau)d\tau$$

does not belong to the class $D(R^1)$ due to $\operatorname{supp}\psi(t) = (-\infty, \tau_1]$, where $\tau_1 = \sup(\operatorname{supp}\phi(\tau))$. But equality (2) is correct because we suppose that $\operatorname{supp} x(t) \subset [0, +\infty)$. Hence set $\operatorname{supp} x(t) \cap \operatorname{supp}\psi(t)$ is bounded.

In that case we can replace function $\psi(t)$ with the finite function

$$\psi_1(t) = \int_t^\infty K_1(\tau,t)\phi(\tau)d\tau,$$

where $K_1(\tau,t) \in D(R^2)$, $K_1(\tau,t) = K(\tau,t)$ on the set $\{(\tau,t)|t,\tau \in [0,\tau_1]\}$. Then on the set $\psi_1(t) = \psi(t)$ the value of the function $(x(t), \psi_1(t))$ is defined. This value does not depend on selection of the function $K_1(\tau,t)$ outside the stated set. The functional $(\Theta(t) * K(\tau,t)x(t))|_{\tau=t}$ belongs to D'_+. In fact, the linearity follows from the properties of linearity of integral and functional $x(t) \in D'_+$. Let us prove the continuity. If $\phi_k(\tau) \to 0$ in $D(R^1)$, then $\exists R > 0 : \operatorname{supp}\phi_k(\tau) \subset [-R, R]$ for any $k \in N$. Let $K_1(\tau,t) \in D(R^2)$, $K_1(\tau,t) = K(\tau,t)$ on the set $\{(\tau,t)|t,\tau \in [0,R]\}$. Then the sequence

$$\psi_1^k(t) = \int_t^\infty K_1(\tau,t)\phi_k(\tau)d\tau \to 0 \text{ in } D(R^1).$$

From this follows $(x(t), \psi_1^k(t)) \to 0$. If $\operatorname{supp}\phi(\tau) \subset (-\infty, 0)$, then $\operatorname{supp}\psi(t) \subset (-\infty, 0)$ and $\operatorname{supp} x(t) \cap \operatorname{supp}\psi(t) = \emptyset$. Hence

$$\operatorname{supp}\big((\Theta(t) * K(\tau,t)x(t)|_{\tau=t}\big) \subset [0, +\infty).$$

Remark 1. If $x(t) \in D'_+$ is regular generalized function, i.e. $x(t) =$

$u(t)\Theta(t)$, where $u(t)$ is locally integrable then

$$\big((\Theta(t) * K(\tau,t)x(t))\big|_{\tau=t}, \phi(t)\big) = \int\limits_0^{\infty} u(t) \int\limits_t^{\infty} K(\tau,t)\phi(\tau)d\tau dt$$

$$= \int\limits_0^{\infty}\Big(\int\limits_0^{\tau} K(\tau,t)u(t)dt\Big)\phi(\tau)d\tau = \Big(\Big(\int\limits_0^{t} K(t,s)u(s)ds\Big)\Theta(t), \phi(t)\Big),$$

i.e.

$$\big(\Theta(t) * K(\tau,t)x(t)\big)\big|_{\tau=t} = \int\limits_0^{t} K(t,s)u(s)ds\Theta(t).$$

Remark 2. If $x(t) = \delta^{(m)}(t)$, then

$$\big(\Theta(t) * K(\tau,t)\delta^{(m)}(t)\big)\big|_{\tau=t} = (-1)^m \frac{\partial^m K(t,0)}{\partial s^m}\Theta(t)$$

$$+ \sum_{i=0}^{m-1}(-1)^i \sum_{l=0}^{i} C_{m-1-l}^{i-l} C_m^l \frac{\partial^i K(0,0)}{\partial t^{i-l}\partial s^l}\delta^{(m-1-i)}(t). \tag{3}$$

In fact,

$$\Big(\big(\Theta(t) * K(\tau,t)\delta^{(m)}(t)\big)\big|_{\tau=t}, \phi(t)\Big) \overset{\text{def}}{=} (-1)^m \Big(\frac{d^m}{dt^m}\int\limits_t^{+\infty} K(\tau,t)\phi(\tau)d\tau\Big)\Big|_{t=0}$$

$$= (-1)^m \Big(\int\limits_t^{+\infty} \frac{\partial^m}{\partial t^m}K(\tau,t)\cdot\phi(\tau)d\tau - \sum_{j=1}^{m}\frac{d^{m-j}}{dt^{m-j}}\Big(\Big(\frac{\partial^{j-1}K(\tau,t)}{\partial t^{j-1}}\phi(\tau)\Big)\Big|_{\tau=t}\Big)\Big)\Big|_{t=0} =$$

$$= (-1)^m \int\limits_0^{+\infty} K_{0m}(\tau,0)\phi(\tau)d\tau$$

$$+(-1)^{m+1}\sum_{i=0}^{m-1}\sum_{j=1}^{i+1} C_{m-j}^{m-1-i}\Big(\frac{d^{i+1-j}}{dt^{i+1-j}}K_{0\,j-1}(t,t)\Big)\Big|_{t=0}\cdot\phi^{(m-1-i)}(0)$$

$$= (-1)^m \int\limits_0^{+\infty} K_{0m}(\tau,0)\phi(\tau)d\tau$$

$$+(-1)^{m+1}\sum_{i=0}^{m-1}\sum_{j=0}^{i} C_{m-j-1}^{m-1-i}\Big(\frac{d^{i-j}}{dt^{i-j}}K_{0\,j}(t,t)\Big)\Big|_{t=0}\cdot(-1)^{m-1-i}\big(\delta^{(m-1-i)}(t), \phi(t)\big),$$

but

$$\sum_{j=0}^{i} C_{m-j-1}^{m-1-i}\Big(\frac{d^{i-j}}{dt^{i-j}}K_{0\,j}(t,t)\Big)\Big|_{t=0} = \sum_{l=0}^{i} C_{m-1-l}^{i-l} C_m^l K_{i-l\,l}(0;0).$$

Now let us come back to (1). Let $x(t) \in C_{[0,t_0]}$ is the solution of (1). If function $x(t)$ is continued by zero for $t < 0$, then function $x(t)$ is the generalized solution of equation

$$(\Theta(t) * K(\tau,t)x(t)\Theta(t))|_{\tau=t} = f(t)\Theta(t).$$

Definition (2) is used here.

We call the problem of construction of the solution $x(t) \in D'_+$ of the equation

$$(\Theta(t) * K(\tau,t)x(t))|_{\tau=t} = f(t)\Theta(t), \tag{4}$$

as problem of solvability of initial (1) in the class D'_+.

3. Generalized Solutions Construction

Now we introduce the basic condition to be used below:

A) $K_{t^i}^{(i)}(t,s)|_{s=t} = 0,\ i = 0, 1, ..., n-1,$
$K_{t^n}^{(n)}(t,s)|_{s=t} \sim\ at^m,\ a \neq 0$ for $t \to 0, m \geq 0.$

Taylor formula gives us $K(t,s) = (t-s)^n Q(t,s),\ K_{t^n}^{(n)}(t,s)|_{s=t} = n!Q(t,t)$, where

$$Q(t,s) = Q_1(t,s) + \sum_{i+k\geq m} a_{ik} t^i s^k,$$

$$Q_1(t,t) = 0,\ \sum_{i+k=m} a_{ik} = a.$$

If $K(t,s)$ is not the convolution, then we can consider the most interesting case $m \geq 1$. In this case the conditions of existence and uniqueness of the generalized solutions of the (1) are not well studied. We follow paper [9] and look for the solution as the following series

$$x = c_0\delta(t) + ... + c_n\delta^{(n)}(t) + u(t)\Theta(t), \tag{5}$$

where $\delta(t)$ is the Dirac function and $u(t)$ is regular function.

On the base of formula (3) due to the condition **A** for $j \leq n$ the following equalities are correct

$$(\Theta(t) * K(\tau,t)\delta^{(j)}(t))|_{\tau=t} = (-1)^j K_{s^j}^{(j)}(t,0)\Theta(t).$$

From these equalities follows that regular item $u(t)$ should satisfy the equation

$$\int\limits_0^t K(t,s)u(s)ds = F(t,c), \tag{6}$$

where

$$F(t,c) = f(t) - \sum_{j=0}^{n} (-1)^j K_{s^j}^{(j)}(t,0)c_j. \tag{7}$$

Let vector $c = (c_0, ..., c_n)'$ satisfies the equalities

$$\sum_{j=0}^{n} (-1)^j \frac{\partial^{i+j} K(0,0)}{\partial t^i \partial s^j} c_j = f^{(i)}(0), \ i = 0, 1, ..., n. \tag{8}$$

Then (6) is an equivalent of the Volterra integral equation of the third kind:

$$n!Q(t,t)u(t) + \int\limits_0^t \frac{\partial^{n+1} K(t,s)}{\partial t^{n+1}} u(s)ds = F^{(n+1)}(t,c), \tag{9}$$

where $Q(t,t) \sim \ at^m$. For existence of the regular solution $u(t)$ of (9) due to condition **A** it is necessary the equality $F_{t^{n+1}}^{(n+i)}(0,c) = 0, i = 1, ..., m$ to be hold.

That is why the sought vector c should satisfy the following system

$$\sum_{j=0}^{n} (-1)^j \frac{\partial^{i+j} K(0,0)}{\partial t^i \partial s^j} c_j = f^{(i)}(0), \ i = 0, 1, ..., n+m. \tag{10}$$

If system (10) is not solvable, then (1) does not have the generalized solutions (5) with singularity order n.

Lemma 1. *Let the following conditions hold true:*

$$\frac{\partial^{n+m} K(0,0)}{\partial t^i \partial s^{m+n-i}} \neq 0, \ i = m, m+1, ..., m+n,$$

$Q_1(t,s) = 0$, *condition A and* $f^{(i)}(0) = 0$, $i = 0, 1, ..., m-1$. *Then system (10) has unique solution.*

Proof. To prove this lemma it is enough to note that system (10) in the conditions of this lemma is following

$$\sum_{j=0}^{n} (-1)^j \frac{\partial^{m+k+j} K(0,0)}{\partial t^{m+k} \partial s^j} c_j = f^{(m+k)}(0), \ k = 0, 1, ..., n. \tag{11}$$

In this system the matrix is low triangular and

$$|det\triangle| = \prod_{i=0}^{n} |\frac{\partial^{m+n} K(0,0)}{\partial t^{m+i} \partial s^{n-i}}| \neq 0.$$

Let vector c satisfy the system (10). Then, for all t, we have

$$\sum_{i=0}^{m+n} \left(\sum_{j=0}^{n} \frac{\partial^{i+j} K(0,0)}{\partial t^i \partial s^j} c_j - f^{(i)}(0) \right) \frac{t^i}{i!} = 0$$

and we can rewrite the right hand side of (6) as follows:

$$F(t,c) = f(t) - \sum_{i=0}^{n+m} f^{(i)}(0) \frac{t^i}{i!} - \sum_{j=0}^{n} (-1)^j \left(\frac{\partial^j K(t,0)}{\partial s^j} - \sum_{i=0}^{n+m} \frac{\partial^{i+j} K(0,0)}{\partial t^i \partial s^j} \frac{t^i}{i!} \right) c_j.$$

Due to the foregoing formulae on the base of Taylor formula, we have $F(t,c) = O(t^{n+m+1})$.

Finally,

$$\lim_{t \to 0} \frac{F^{n+1}(t,c)}{Q(t,t)} = 0, \tag{12}$$

if $Q_1(t,s) = 0$.

In addition, due to condition **A** in the area $0 < s \leq t \leq t_0$, we can guarantee the following estimate:

$$\frac{\frac{\partial^{n+1} K(t,s)}{\partial t^{n+1}}}{Q(t,t)} = O\left(\frac{1}{t}\right). \tag{13}$$

Integral (9) has the regular singularity in zero due to estimate (13).

Further, let the homogeneous equation which corresponds to (1) has only zero solution. In this case we can construct the formal solution of integral (9) by the method of unknown coefficients:

$$u(t) \sim \sum_{1}^{\infty} u_i t^i. \tag{14}$$

□

Remark 3. Lemma 1 is still correct if $Q(t,s) = \sum_{i,k} a_{ik} t^i s^k$ where $a_{ik} = 0$ for $i + k \leq m - 1, \ \ i \leq m - 1, \ \ k \leq n$.

Remark 4. Due to condition **A**, for $Q_1(t,s) = 0$ we have

$$\frac{\partial^{n+1} K(t,s)}{\partial t^{n+1}} = \sum_{i+k=m-1} b_{ik} t^i s^k + O((t+s)^m).$$

Then uniqueness of the solution of the homogeneous equation is equivalent of the condition

$$a + \sum_{i+k=m-1} b_{ik} \frac{1}{k+l} \neq 0$$

for $l = 0, 1, ...$

Theorem 1. *Let the homogeneous equation which correspond to (1) has only zero formal solution (14) and the conditions of Lemma 1 hold true. Then (1) has unique solution (5) in the class D'_+.*

Proof. We define the vector c in expansion (5) from system (11) by substituting it in the right hand side of (9). We can find N first coefficients u_i of formal solution (14).

Let $u = \sum\limits_{i=1}^{N} U_i t^i + V(t)$ in (9). Then to define $V(t)$ we get the integral equation

$$V(t) + \int\limits_0^t L(t,s)V(s)ds = b(t),$$

where

$$L(t,s) = \frac{K^{(n+1)}_{t^{n+1}}(t,s)}{n!Q(t,t)},$$

$$b(t) = \frac{F^{(n+1)}(t,c)}{n!Q(t,t)} - \sum_{i=0}^{N} \bigl(U_i t^i + \int_0^t L(t,s)U_i s^i ds\bigr).$$

Taking into account (13), we can note that $b(t) = O(t^{N+1})$. The kernel $\frac{c}{t}$, $c > 0$ has the resolvent $\frac{c}{t}\bigl(\frac{t}{s}\bigr)^c$. Because of estimate (13), the kernel $L(t,s)$ in the area $0 < s \leq t \leq t_0$ for small enough t_0 also has resolvent $R(t,s)$ with similar estimate. But in this case, for an N great enough, the integral $\int_0^t R(t,s)b(s)ds$ is converging and we can define function $V(t)$ by known formula

$$V(t) = b(t) + \int_0^t R(t,s)b(s)ds.$$

□

Remark 5. If the assumptions of Theorem 1 hold true, and $f^{(i)}(0) = 0$, $i = m, ..., m+n$, then $c_0 = ... = c_n = 0$ and solution (5) is classical.

Now we consider the generalized solutions of (1) for $\sum_{i=0}^{m+n} |f^{(i)}(0)| \neq 0$. We will prove that in this case the generalized solutions with the highest singularity order can exist.

We use the following condition below

B) $\frac{\partial^{i+j} K(0,0)}{\partial t^i \partial s^j} = \begin{cases} 0, \ 0 \le i+j \le n+m-1 \\ \neq 0, \ i+j = n+m. \end{cases}$

If condition **A** holds true, then

$$Q(t,s) = \sum_{i+k \ge m} a_{ik} t^i s^k.$$

We look for the solution of (1) as following

$$x(t) = c_0 \delta(t) + ... + c_{n+m} \delta^{(n+m)}(t) + u(t)\Theta(t). \tag{15}$$

We can define vector $(c_0, ..., c_{n+m})'$ from the system

$$\Xi c = \beta, \tag{16}$$

where

$$\Xi = \left\| (-1)^j \frac{\partial^{i+j} K(0,0)}{\partial t^i \partial s^j} \right\|_{i,j=0,\overline{n+m}},$$

$$\beta = (f(0), f'(0), ..., f^{n+m}(0)).$$

Due to condition **B**, the matrix of system (10) is lower triangular and non degenerated. To define the regular component $u(t)$ we again have (6) where $F(t,c) = f(t) - \sum_{j=0}^{n+m} (-1)^j K^{(j)}_{s^j}(t,0) c_j$.

The solution c of system (16) for any t obviously satisfies the equality

$$\sum_{j=0}^{n+m} (-1)^j \sum_{i=n+m-j}^{n+m} \frac{\partial^{i+j} K(0,0)}{\partial t^i \partial s^j} \frac{t^i}{i!} c_j = \sum_{i=0}^{n+m} f^{(i)}(0) \frac{t^i}{i!}.$$

From condition **A**, $\frac{F^{n+1}(t,c)}{Q(t,t)} = O(t)$ and from the aforesaid proof of Theorem 1 we have the following theorem.

Theorem 2. *Let the homogeneous equation which corresponds to* (1) *has only zero solution and conditions* A *and* B *hold true. Then, for any* $f(t) \in C^\infty[0, t_0)$, (1) *has unique generalized solution* (15).

Remark 6. If the assumptions of Theorem 2 hold true and $f(0) = f'(0) = ... = f^{(m-1)}(0) = 0$, then $c_{n+1} = ... = c_{n+m} = 0$ and the result of Theorem 1 is obtained.

4. Notes on the Families of Parametric Generalized Solutions

Let the assumptions of Theorem 2 hold true and some of $[n+m]$th derivatives of the kernel $K(t,s)$ at the point (0,0) be zeros. Then matrix Ξ in system (16) is degenerate. If in this case system (16) remains solvable, then (1) has $n+m+1-r$-parametric family of generalized solutions (15), where $r = \mathrm{rank}\Xi$. If in this case we assume that the homogeneous equation of (1) has d nontrivial solutions for $d \leq n+m+1-r$, then d arbitrary parameters in vector c can be defined by the construction of the formal series (14). But in this case the coefficients u_i of the formal series (14) remains arbitrary and we again get $n+m+1-r$ parametric family of generalized solutions (15).

Note also that if system (15) is not solvable then there are no generalized solutions of (1).

Conclusion. In our paper we have presented new approach to the construction of the generalized solutions of the Volterra integral equations of the first kind. This approach along with the methods presented in paper [4] and in the monograph [8] (chapter 6) provides a base for construction of the theory of generalized solutions of the Volterra integral equations of the first kind in the Banach spaces.

On the base of Theorems 1 and 2 one can construct generalized solutions of the Volterra equation (1) based on a two-stage analytical-numerical scheme. The solutions contain two components: the singular and the regular one. These components have to be constructed separately. The order of singular component is discussed above. On the first stage we construct the singular component of the solution by solving the special linear algebraic system. On the second we construct the regular component of generalized solution by solving integral (6) with transformed right hand side. Here, for example, we can use regularized numerical methods [1].

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Institute of Mathematics, Economics and Informatics, Irkutsk State University, 1 K. Marks Str, 664003, Irkutsk, Russia

E-mail address: sidorov@math.isu.runnet.ru

Energy Systems Institute of Siberian Branch of Russian Academy of Sciences, 130 Lermontov Str., 664033, Irkutsk, Russia

E-mail address: dsidorov@isem.sei.irk.ru

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