A lower semicontinuity result in SBD for surface integral functionals of Fracture Mechanics.

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Abstract

A lower semicontinuity result is proved in the space of special functions of bounded deformation for a fracture energetic model

$$\int_{J_u} \Psi([u], \nu_u) d\mathcal{H}^{N-1} \ , \ [u] \cdot \nu_u \geq 0 \ \mathcal{H}^{N-1} - \text{ a. e. on } J_u.$$

A characterization of the energy density Ψ, which ensures lower semicontinuity, is also given.

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1 Introduction

In recent years a considerable attention has been devoted to Fracture Mechanics in order to describe the behavior of brittle bodies, such as the propagation of the fracture inside the material, the equilibrium configurations, the crack site, the interaction between the elastic energy and the dissipated energy. In particular, in [5, 6] it has been studied, both from the mechanical and computational view point, in the regime of linearized elasticity, the propagation of the fracture in a cracked body with a dissipative energy a la Barenblatt, i.e. of the type $\int_K \phi([u]\cdot\nu_u, [u]\cdot\tau_u) d\mathcal{H}^{N-1}$, where K denotes the unknown crack site, $[u]\cdot\nu_u$, $[u]\cdot\tau_u$ represent the detachment and the sliding components respectively, of the opening of the fracture $[u]$, and the energy density ϕ has the form $\phi([u]\cdot\nu_u, [u]\cdot\tau_u) = \begin{cases} 0 & \text{if } [u]\cdot\nu_u = [u]\cdot\tau_u = 0, \\ \text{constant} & \text{if } [u]\cdot\nu_u \geq 0, \\ +\infty & \text{if } [u]\cdot\nu_u < 0 \end{cases}$

It has to be emphasized that the form of the energy density ϕ also takes into account an infinitesimal noninterpenetration constraint, i.e. all the deformations u pertainining to the effective description of the energy must satisfy $[u]\cdot\nu_u \geq 0$ $\mathcal{H}^{N-1}$ a.e. on K.

In order to derive, from the mathematical view point, the properties of the energy ϕ above which guarantee lower semicontinuity with respect to the natural convergences (2.13) $\div$ (2.15) below, in order to generalize the models contained in [5, 6] and to extend the lower semicontinuity results for surface integrals contained in [8], the following result has been proved in [11]:

Theorem 1.1. *Let Ω be a bounded open subset of $\mathbb{R}^N$, Let*

$$\Phi := \{\varphi : [0, +\infty[\to [0, +\infty[, \varphi \text{ convex, subadditive and nondecreasing}\} \tag{1.1}$$

and let $\varphi \in \Phi$. Let $\{u_h\}$ be a sequence in $SBD(\Omega)$, such that $[u_h]\cdot\nu_{u_h} \geq 0$ $\mathcal{H}^{N-1}$-a.e. on J_{u_h} for every h, converging to u in $L^1(\Omega;\mathbb{R}^N)$ satisfying (2.12) below, with a function $\theta : [0, +\infty[\to [0, \infty[$ nondecreasing and verifying the superlinearity condition (2.11) below. Then

$$[u]\cdot\nu_u \geq 0 \ \mathcal{H}^{N-1} - \text{ a.e. on } J_u, \tag{1.2}$$

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and

$$\int_{J_u} \varphi([u]\cdot\nu_u)d\mathcal{H}^{N-1} \leq \liminf_{h\to+\infty}\int_{J_{u_h}} \varphi([u_h]\cdot\nu_{u_h})d\mathcal{H}^{N-1}. \tag{1.3}$$

Clearly the class Φ in (1.1) includes functions of the type ϕ above, but it has also to be observed that, in general, the functions in Φ can be truly convex. In fact, typical examples of functions in Φ are given by $\varphi : s \in \mathbb{R}^+ \mapsto (1+s^p)^{\frac{1}{p}}$, $p \geq 1$, but in practice this class of functions does not perfectly fit the mechanical framework, where actually a 'concave-type' behavior is expected.

The aim of this paper, indeed, consists of finding a mechanically more interesting class of functions containing the function ϕ in [5, 6], and also including energy densities with a more general dependence from the opening of the fracture $[u]$ and from the normal of the crack site ν_u, rather than just from their scalar product $[u]\cdot\nu_u$. To this end we introduce the following type of functions. Let $\Psi : (a,b) \in \mathbb{R}^N \times S^{N-1} \to [0,+\infty[$ be defined as follows

$$\Psi : (a,b) \in \mathbb{R}^N \times S^{N-1} \mapsto \sup_{\xi\in S^{N-1}} |b\cdot\xi|\psi(|a\cdot\xi|), \tag{1.4}$$

where $\psi : [0,+\infty[\to [0,+\infty[$ is a lower semicontinuous, nondecreasing subadditive function (more generally a lower semicontinuos function such that $\psi(|\cdot|)$ is subadditive). Consequently the following lower semicontinuity result with respect to convergences (2.13) $\div$ (2.15) can be established:

Theorem 1.2. *Let Ω be a bounded open subset of $\mathbb{R}^N$, let $\theta : [0,+\infty[\to [0,+\infty[$ be a non-decreasing function verifying the superlinearity condition (2.11), and let Ψ be as in (1.4) where $\psi : [0,+\infty[\to [0,+\infty[$ is a lower semicontinuous function such that $t \in]-\infty,+\infty[\to \psi(|t|)$ is subadditive. Let $\{u_h\}$ be a sequence in $SBD(\Omega)$ satisfying the bound (2.12), such that $[u_h]\cdot\nu_{u_h} \geq 0$ $\mathcal{H}^{N-1}$-a.e. on J_{u_h} for every h and converging to u in $L^1(\Omega;\mathbb{R}^N)$. Then (1.2) holds and*

$$\int_{J_u} \Psi([u],\nu_u)d\mathcal{H}^{N-1} \leq \liminf_{h\to+\infty}\int_{J_{u_h}} \Psi([u_h],\nu_{u_h})d\mathcal{H}^{N-1} \tag{1.5}$$

We observe that the energy $\int_{J_u}\phi([u]\cdot\nu_u,[u]\cdot\tau_u)d\mathcal{H}^{N-1}$ in [5, 6] with ϕ as above can be recasted in the terms of a suitable Ψ as in (1.4) requiring that the noninterpenetration constraint (1.2) is verified, (see (i) of Examples 4.7). On the other hand, as observed in Remark 4.8 the class of functions of the type Ψ is different from Φ in (1.1), and not including it. The difference among the two classes is not very surprising, and in fact, also the techniques adopted to prove the two lower semicontinuity results (Theorem 1.1 and Theorem 1.2) are very different, the first relying on Measure Theory and the second on the structure of the Special functions of Bounded Deformation, enlightened in [3, 8]. A rather comprehensive analysis of the properties and a characterization of the function Ψ in (1.4) are given below. In detail, the paper is organized as follows. In Section 2 the principal results from Measure Theory, concerning spaces of functions with bounded deformation, are recalled. Section 3 is devoted to the proof of the lower semicontinuity theorem, while the properties of the energy density are investigated in Section 4, where also a comparison with the functions belonging to the class (1.1) is given.

2 Notations and Preliminaries

Here and in the sequel, let Ω be a bounded open subset of $\mathbb{R}^N$. We shall usually suppose, when non explicitly mentioned, (essentially to avoid trivial cases) that $N > 1$. Let $u \in L^1(\Omega;\mathbb{R}^m)$, the set of Lebesgue points of u is denoted by Ω_u. In other words $x \in \Omega_u$ if and only if there exists $\tilde{u}(x) \in \mathbb{R}^m$ such that

$$\lim_{\varrho\to 0^+}\frac{1}{\varrho^N}\int_{B_\varrho(x)} |u(y)-\tilde{u}(x)|dy = 0.$$

The space $BD(\Omega)$ of *vector fields with bounded deformation* is defined as the set of vector fields $u = (u^1,\dots,u^N) \in L^1(\Omega;\mathbb{R}^N)$ whose distributional gradient $Du = \{D_iu^j\}$ has the symmetric part

$$Eu = \{E_{ij}u\}, E_{ij}u = (D_iu^j + D_ju^i)/2$$

which belongs to $\mathcal{M}_b(\Omega; M^{N\times N}_{sym})$, the space of bounded Radon measures in Ω with values in $M^{N\times N}_{sym}$, the space of symmetric $N\times N$ matrices. For $u \in BD(\Omega)$, the *jump set* J_u is defined as the set of points $x \in \Omega$ where u has two different *one sided Lebesgue limits* $u^+(x)$ and $u^-(x)$, with respect a suitable direction $\nu_u(x) \in S^{N-1} = \{\xi \in \mathbb{R}^N : |\xi| = 1\}$, i.e.

$$\lim_{\varrho\to 0^+} \frac{1}{\varrho^N} \int_{B^\pm_\varrho(x,\nu_u(x))} |u(y) - u^\pm(x)|dy = 0, \tag{2.1}$$

where $B^\pm_\varrho(x, \nu_u(x)) = \{y \in \mathbb{R}^N : |y - x| < \varrho, (y - x) \cdot \nu_u(x) > 0\}$. In [3] it has been proved that for every $u \in BD(\Omega)$ the jump set J_u is Borel measurable and countably $(\mathcal{H}^{N-1}, N-1)$ rectifiable and $\nu_u(x)$ is normal to the approximate tangent space to J_u at x for $\mathcal{H}^{N-1}$-a.e. $x \in J_u$, where $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure (see [4] and [10]).
Let $u \in BD(\Omega)$, the Lebesgue decomposition of Eu is written as

$$Eu = E^a u + E^s u$$

with $E^a u$ the absolutely continuous part and $E^s u$ the singular part with respect to the Lebesgue measure $\mathcal{L}^N$.
The density of $E^a u$ with respect to $\mathcal{L}^N$ is denoted by $\mathcal{E}u$, i.e. $E^a u = \mathcal{E}u\mathcal{L}^N$. We recall that $E^s u$ can be further decomposed as

$$E^s u = E^j u + E^c u$$

with $E^j u$, the *jump part* of Eu, i.e. the restriction of $E^s u$ to J_u and $E^c u$ the *Cantor part* of Eu, i.e. the restriction of $E^s u$ to $\Omega \setminus J_u$. Furthermore, in [3] it has been proved that

$$E^j u = (u^+ - u^-) \odot \nu_u \mathcal{H}^{N-1} \lfloor J_u \tag{2.2}$$

where $\odot$ denotes the symmetric tensor product, defined by $a \odot b := (a \otimes b + b \otimes a)/2$ for every $a, b \in \mathbb{R}^N$, and $\mathcal{H}^{N-1} \lfloor J_u$ denotes the restriction of $\mathcal{H}^{N-1}$ to J_u, i.e. $(\mathcal{H}^{N-1} \lfloor J_u)(B) = \mathcal{H}^{N-1}(B \cap J_u)$ for every Borel set $B \subseteq \Omega$. Moreover in [3] it has been also proved that $|E^c u|(B) = 0$ for every Borel set $B \subseteq \Omega$ such that $\mathcal{H}^{N-1}(B) < +\infty$, where $|\cdot|$ stands for the total variation. In the sequel, for every $u \in L^1_{loc}(\Omega; \mathbb{R}^N)$ we denote by $[u]$ the vector $u^+ - u^-$. For any $y, \xi \in \mathbb{R}^N$, $\xi \neq 0$, and any $B \in \mathcal{B}(\Omega)$ we define

$$\begin{aligned} &\pi_\xi := \{y \in \mathbb{R}^N : y \cdot \xi = 0\},\\ &B^\xi_y := \{t \in \mathbb{R} : y + t\xi \in B\},\\ &B^\xi := \{y \in \pi_\xi : B^\xi_y \neq \emptyset\}, \end{aligned} \tag{2.3}$$

i.e. π_ξ is the hyperplane orthogonal to ξ , passing through the origin and $B^\xi = p_\xi(B)$, where p_ξ, denotes the orthogonal projection onto π_ξ. B^ξ_y is the one-dimensional section of B on the straight line passing through y in the direction of ξ.

Given a function $u : B \to \mathbb{R}^N$, defined on a subset B of $\mathbb{R}^N$, for every $y, \xi \in \mathbb{R}^N$, $\xi \neq 0$, the function $u^\xi_y : B^\xi_y \to \mathbb{R}$ is defined by

$$u^\xi_y(t) := u^\xi(y + t\xi) = u(y + t\xi) \cdot \xi \text{ for all } t \in B^\xi_y. \tag{2.4}$$

In [3] it has been proved that a vector field u belongs to $BD(\Omega)$ if and only if its '*projected sections*' u^ξ_t belong to $BV(\Omega^\xi_t)$. More precisely the following Structure Theorem (cf. Structure Theorem 4.5 in [3]) has been proved.

Theorem 2.1. *Let $u \in BD(\Omega)$ and let $\xi \in \mathbb{R}^N$ with $\xi \neq 0$. Then*

(i) $E^a u\xi \cdot \xi = \int_{\Omega^\xi} D^a u^\xi_y d\mathcal{H}^{N-1}(y), |E^a u\xi \cdot \xi| = \int_{\Omega^\xi} |D^a u^\xi_y| d\mathcal{H}^{N-1}(y).$

(ii) For $\mathcal{H}^{N-1}$-almost every $y \in \Omega^\xi$, the functions u^ξ_y and $\tilde{u}^\xi_y$ belong to $BV(\Omega^\xi_y)$ and coincide $\mathcal{L}^1$-almost everywhere on Ω^ξ_y, the measures $|Du^\xi_y|$ and $V\tilde{u}^\xi_y$ coincide on Ω^ξ_y ,and $\mathcal{E}u(y+t\xi)\xi\cdot\xi = \nabla u^\xi_y(t) = (\tilde{u}^\xi_y)'(t)$ for $\mathcal{L}^1$-almost every $t \in \Omega^\xi_y$.

(iii) $E^j u\xi\cdot\xi = \int_{\Omega^\xi} D^j u^\xi_y d\mathcal{H}^{N-1}(y)$, $|E^j u\xi\cdot\xi| = \int_{\Omega^\xi} |D^j u^\xi_y| d\mathcal{H}^{N-1}(y)$.

(iv) $(J^\xi_u)^\xi_y = J_{u^\xi_y}$ *for* $\mathcal{H}^{N-1}$*-almost every* $y\in\Omega^\xi$ *and for every* $t\in(J^\xi_u)^\xi_y$

$$\begin{array}{l} u^+(y+t\xi)\cdot\xi = (u^\xi_y)^+(t) = \lim_{s\to t^+}\tilde{u}^\xi_y(s) \\ u^-(y+t\xi)\cdot\xi = (u^\xi_y)^-(t) = \lim_{s\to t^-}\tilde{u}^\xi_y(s), \end{array}$$

where the normals to J_u *and* $J_{u^\xi_y}$ *are oriented so that* $\nu_u\cdot\xi\geq 0$ *and* $\nu_{u^\xi_y} = 1$.

(v) $E^c u\xi\cdot\xi = \int_{\Omega^\xi} D^c u^\xi_y d\mathcal{H}^{N-1}(y), |E^c u\xi\cdot\xi| = \int_{\Omega^\xi} |D^c u^\xi_y| d\mathcal{H}^{N-1}(y)$.

The space $SBD(\Omega)$ of *special vector fields with bounded deformation* is defined as the set of all $u\in BD(\Omega)$ such that $E^c u = 0$, or, in other words

$$Eu = \mathcal{E}u\mathcal{L}^N + (u^+ - u^-)\odot\nu_u\mathcal{H}^{N-1}\lfloor J_u$$

We also recall that if $\Omega\subset\mathbb{R}$, then the space $SBD(\Omega)$ coincides with the space with the space of real valued special functions of bounded variations $SBV(\Omega)$, consisting of the functions whose distributional gradient is a Radon measure with no Cantor part (see [4] for a comprehensive treatment of the subject).

Proposition 2.2. *Let* $u\in BD(\Omega)$ *and let* $\xi_1,\dots,\xi_N$ *be a basis of* $\mathbb{R}^N$*. Then the following three conditions are equivalent:*

(i) $u\in SBD(\Omega)$.

(ii) For every $\xi = \xi_i+\xi_j$ *with* $1\leq i,j\leq n$*, we have* $u^\xi_y\in SBV(\Omega^\xi_y)$ *for* $\mathcal{H}^{N-1}$*-almost every* $y\in\Omega^\xi$.

(iii) The measure $|E^s u|$ *is concentrated on a Borel set* $B\subset\Omega$ *which is* σ*-finite with respect to* $\mathcal{H}^{N-1}$.

Definition 2.3. *For any* $u\in BD(\Omega)$ *we define the non-negative Borel measure* λ_u *on* Ω *as*

$$\lambda_u(B) := \frac{1}{2\omega_{N-1}}\int_{S^{N-1}}\lambda^\xi_u(B)d\mathcal{H}^{N-1}(\xi)\ \ \forall B\in\mathcal{B}(\Omega), \tag{2.5}$$

where, for every $\xi\in S^{N-1}$

$$\lambda^\xi_u(B) := \int_{\Omega^\xi}\mathcal{H}^0(J_{u^\xi_y}\cap B^\xi_y)d\mathcal{H}^{N-1}(y)\ \ \forall B\in\mathcal{B}(\Omega). \tag{2.6}$$

Let

$$J^\xi_u := \left\{x\in J_u : (u^+ - u^-)\cdot\xi\neq 0\right\}, \tag{2.7}$$

we recall that

$$\mathcal{H}^{N-1}(J_u\setminus J^\xi_u) = 0 \text{ for } \mathcal{H}^{N-1} - \text{a.e. } \xi\in S^{N-1}. \tag{2.8}$$

The following result is a consequence of the Structure Theorem

Theorem 2.4. *For every* $u\in BD(\Omega)$ *and any* $\xi\in S^{N-1}$,

$$\lambda^\xi_u(B) = \int_{J^\xi_u\cap B}|\nu_u\cdot\xi|d\mathcal{H}^{N-1}\ \ \forall B\in\mathcal{B}(\Omega), \tag{2.9}$$

where ν_u *is the approximate unit normal to* J_u*. Moreover* $\lambda_u = \mathcal{H}^{N-1}\lfloor J_u$.

The same argument of Theorem 2.4, i.e. (iv) of Theorem 2.1 and the fact that the $(N-1)$-dimensional area factor of p_ξ on J_u is $|\nu_u\cdot\xi|$ guarantees that for every Borel function $g:\Omega\to[0,+\infty]$, it results

$$\int_{J^\xi_u\cap B} g(y)|\nu_u\cdot\xi|d\mathcal{H}^{N-1}(y) = \int_{\Omega^\xi}\int_{p_\xi(J^\xi_u\cap B)} g(y+t\xi)d\mathcal{H}^0(t)d\mathcal{H}^{N-1}(y) \tag{2.10}$$

for any $\xi\in S^{N-1}$.

We recall the following compactness result for sequences in SBD proved in [8], (cf. Theorem 1.1 and Remark 2.3 therein).

Theorem 2.5. *Let $\theta : [0,+\infty[\to [0,+\infty[$ be a non-decreasing function such that*

$$\lim_{t\to 0^+} \frac{\theta(t)}{t} = +\infty. \tag{2.11}$$

Let $\{u_h\}$ be a sequence in $SBD(\Omega)$ such that

$$\|u_h\|_{L^\infty(\Omega;\mathbb{R}^N)} + \int_\Omega \theta(|\mathcal{E}u_h|)dx + \mathcal{H}^{N-1}(J_{u_h}) \leq K \tag{2.12}$$

for some constant K independent of h. Then there exists a subsequence, still denoted by $\{u_h\}$, and a function $u \in SBD(\Omega)$ such that

$$u_h \to u \text{ strongly in } L^1_{loc}(\Omega;\mathbb{R}^N), \tag{2.13}$$

$$\mathcal{E}u_h \rightharpoonup \mathcal{E}u \text{ weakly in } L^1(\Omega; M^{N\times N}_{sym}), \tag{2.14}$$

$$E^j u_h \rightharpoonup E^j u \text{ weakly* in } \mathcal{M}_b(\Omega; M^{N\times N}_{sym}), \tag{2.15}$$

$$\mathcal{H}^{N-1}(J_u) \leq \liminf_{h\to+\infty} \mathcal{H}^{N-1}(J_{u_h}) \tag{2.16}$$

The following results from Measure Theory will be exploited in the sequel, (cf. Lemma 2.35 in [4] and Lemma 3.6 in [9] respectively).

Lemma 2.6. *Let λ be a positive σ-finite Borel measure in Ω and let $\varphi_i : \Omega \to [0,\infty]$, $i \in \mathbb{N}$, be Borel functions. Then*

$$\int_\Omega \sup_i \varphi_i d\lambda = \sup\left\{\sum_{i\in I}\int_{A_i} \varphi_i d\lambda\right\}$$

where the supremum ranges over all finite sets $I \subset \mathbb{N}$ and all families $\{A_i\}_{i\in I}$ of pairwise disjoint open sets with compact closure in Ω.

Lemma 2.7. *Let k be a positive integer, and let*

$$\mathcal{M}_k = \{\lambda \in \mathcal{M}(\Omega;\mathbb{R}^k) : \mathcal{H}^0(\mathrm{spt}\lambda) \leq k\}.$$

Then $\mathcal{M}_k$ is sequentially weakly $*$ closed. Moreover, for every lower semicontinuous function $g : \Omega\times\mathbb{R}^n \to [0,+\infty]$ satisfying the subadditivity condition

$$g(x, s_1 + s_2) \leq g(x, s_1) + g(x, s_2) \text{ for every } x \in \Omega \text{ and } s_1, s_2 \in \mathbb{R}^n,$$

the functional

$$G(\lambda) = \int_\Omega g(x, \lambda(x))d\mathcal{H}^0 = \sum_{x\in\mathrm{spt}\lambda} g(x, \lambda(\{x\}))$$

is sequentially weakly $*$ lower semicontinuous on $\mathcal{M}_k$.

Finally we recall an approximation result that will be used in the next section (see Lemma 1.61 in [4]).

Lemma 2.8. *Let $c \in \mathbb{R}, u : X \to [c,\infty]$ not identically equal to ∞ and define for $t > 0$*

$$u_t := \inf\{u(y) + td(x,y) : y \in X\},$$

where d is the distance in X. Then $\mathrm{Lip}(u_t) \leq t, u_t \leq u$ ($\mathrm{Lip}(v)$ being the Lipschitz constant of the function v) and $u_t(x) \uparrow u(x)$ as $t \uparrow \infty$ whenever x is a lower semicontinuity point of u.

3 Lower Semicontinuity

This section is devoted to the proof of Theorem 1.2. To this end we state and prove some preliminary lower semicontinuity results.

Lemma 3.1. *Let $\psi : [0,+\infty[\to [0,+\infty[$ be a lower semicontinuous function such that $\psi(|\cdot|)$ is subadditive. Let $\theta : [0,+\infty[\to [0,+\infty[$ be a nondecreasing function such that the superlinearity condition (2.11) holds. Let I be an open interval of $\mathbb{R}$. Let $\{u_j\} \subset SBV(I)$ such that*

$$\|u_j\|_{L^\infty(I)} + \int_I \theta(|u_j'|)dx + \mathcal{H}^0(J_{u_j}) \leq C$$

(here u_j' denotes the absolutely continuous part of Du_j with respect to the Lebesgue measure). Assume also that $u_j \to u$ in $L^1(I)$. Then

$$\int_{J_u} \psi(|u^+ - u^-|)d\mathcal{H}^0 \leq \liminf_{j\to+\infty} \int_{J_{u_j}} \psi(|u_j^+ - u_j^-|)d\mathcal{H}^0.$$

Proof. First consider a subsequence $\{u_{j_k}\}$ such that $\liminf_{j\to+\infty} \int_{J_{u_j}} \psi(|u_j^+ - u_j^-|)d\mathcal{H}^0$ is a limit on k. By virtue of Theorem 2.5, it results that it admits a further subsequence, still denoted by $\{u_{j_k}\}$, such that all the convergence relations (2.13)÷(2.16) hold in I. Consequently, since $\psi(|\cdot|)$ is subadditive, Lemma 2.7, applied to the measure $\lambda = (u^+ - u^-)\mathcal{H}^0\lfloor J_u$ and to $g(x,s) = \psi(|s|)$, ensures that

$$\int_{J_u} \psi(|u^+ - u^-|)d\mathcal{H}^0 \leq \lim_{k\to+\infty} \int_{J_{u_{j_k}}} \psi(|u_{j_k}^+ - u_{j_k}^-|)d\mathcal{H}^0 = \liminf_{j\to+\infty} \int_{J_{u_j}} \psi(|u_j^+ - u_j^-|)d\mathcal{H}^0.$$

□

Remark 3.2. *We emphasize that a mechanically relevant class of lower semicontinuous functions ψ satisfying the assumptions of Lemma 3.1 is consituted of functions $\psi : [0,+\infty[\to [0,+\infty[$ subadditive and nondecreasing. On the other hand the nondecreasing behavior of ψ is just a sufficient condition to ensure that the function $t \in]-\infty,+\infty[\to \psi(|t|)$ being subadditive: any function ψ with range in $[1,2]$ would still satisfy the hypotheses of Lemma 3.1.*

Finally we observe that the lower semicontinuity result proved in Lemma 3.1 still holds when replacing the open interval I, by any open set of $\mathbb{R}$, thanks to the superadditivity of the liminf operator, at least on non-negative families.

An interesting example of functions ψ lower semicontinuous, nondecreasing and subadditive, clearly satisfying the assumptions of Lemma 3.1 is given by the Dugdale function

$$\psi_D : t \in [0,+\infty[\to \min\{t,1\} \tag{3.1}$$

relevant in the applications to Continuum Mechanics.

The proof of the following result exploits the structure of SBD functions enlightened in Theorem 1.1 in [8].

Lemma 3.3. *Let $\psi : [0,+\infty[\to [0,+\infty[$ be a lower semicontinuous function such that the function $t \in]-\infty,+\infty[\to \psi(|t|)$ is subadditive. Let Ω be a bounded open subset of $\mathbb{R}^N$, and let $\theta : [0,+\infty[\to [0,+\infty[$ be a non-decreasing function verifying the superlinearity condition (2.11). Let $\{u_h\}$ be a sequence in $SBD(\Omega)$ satisfying the bound (2.12), such that $[u_h]\cdot\nu_{u_h} \geq 0$ $\mathcal{H}^{N-1}$-a.e. on J_{u_h} for every h and converging to u in $L^1(\Omega;\mathbb{R}^N)$. Then*

$$\int_{J_u} |\xi\cdot\nu_u|\psi(|[u](y)\cdot\xi|)d\mathcal{H}^{N-1}(y) \leq \liminf_{h\to+\infty} \int_{J_{u_h}} |\xi\cdot\nu_{u_h}|\psi(|[u_h](y)\cdot\xi|)d\mathcal{H}^{N-1}(y) \tag{3.2}$$

for $\mathcal{H}^{N-1}$-a.e. $\xi \in S^{N-1}$.

Proof. Let $\{u_h\} \subset SBD(\Omega)$ satisfying the bound (2.12) and converging to $u \in L^1(\Omega)$. From Theorem 2.5 $u \in SBD(\Omega)$.

Let $\xi \in S^{N-1}$, and let $p_\xi : J_u \to \pi_\xi$ be the orthogonal projection onto π_ξ. First we observe that (iv) in Theorem 2.1 and Proposition 2.2 ensure that for $\mathcal{H}^{N-1}$- a.e. $y \in \Omega^\xi$ it results $([u^\xi_y])(t) = ([u]\cdot\xi)(y+t\xi)$ for every $t \in J_{u^\xi_y}$ and $([{u_h}^\xi_y])(t) = ([u_h]\cdot\xi)(y+t\xi)$ for every $t \in J_{{u_h}^\xi_y}$, with $u^\xi_y, {u_h}^\xi_y \in SBV(\Omega^\xi_y)$ for $\mathcal{H}^{N-1}$-a.e. $y \in \Omega^\xi$.

On the other hand, by (2.7) and (2.8), it results that

$$\begin{aligned}\int_{J_u} |\xi\cdot\nu_u|\psi(|[u](y)\cdot\xi|)d\mathcal{H}^{N-1}(y) &= \int_{J^\xi_u} |\xi\cdot\nu_u|\psi(|[u](y)\cdot\xi|)d\mathcal{H}^{N-1}(y),\\ \int_{J_{u_h}} |\xi\cdot\nu_{u_h}|\psi(|[u_h](y)\cdot\xi|)d\mathcal{H}^{N-1}(y) &= \int_{J^\xi_{u_h}} |\xi\cdot\nu_{u_h}|\psi(|[u_h](y)\cdot\xi|)d\mathcal{H}^{N-1}(y)\end{aligned} \tag{3.3}$$

for every $h \in \mathbb{N}$ and for $\mathcal{H}^{N-1}$-a.e. $\xi \in S^{N-1}$. Formulas (3.3), (2.10) guarantee that there exists $N \subset S^{N-1}$ such that $\mathcal{H}^{N-1}(N) = 0$ and it results:

$$\int_{J_u} |\xi\cdot\nu_u|\psi(|[u](y)\cdot\xi|)d\mathcal{H}^{N-1}(y) = \int_{\Omega^\xi}\Big[\int_{J_{u^\xi_y}} \psi(|[u^\xi_y]|(t))d\mathcal{H}^0(t)\Big]d\mathcal{H}^{N-1}(y),$$

and

$$\int_{J_{u_h}} |\xi\cdot\nu_{u_h}|\psi(|[u_h](y)\cdot\xi|)d\mathcal{H}^{N-1}(y) = \int_{\Omega^\xi}\Big[\int_{J_{{u_h}^\xi_y}} \psi(|[{u_h}^\xi_y]|(t))d\mathcal{H}^0(t)\Big]d\mathcal{H}^{N-1}(y),$$

for every $h \in \mathbb{N}$ and for every $\xi \in S^{N-1}\setminus N$.

Consequently the proof will be completed once we show that

$$\int_{\Omega^\xi}\Big[\int_{J_{u^\xi_y}} \psi(|[u^\xi_y]|(t))d\mathcal{H}^0(t)\Big]d\mathcal{H}^{N-1}(y) \le \liminf_{h\to+\infty}\int_{\Omega^\xi}\Big[\int_{J_{{u_h}^\xi_y}} \psi(|[{u_h}^\xi_y]|(t))d\mathcal{H}^0(t)\Big]d\mathcal{H}^{N-1}(y) \tag{3.4}$$

for every $\xi \in S^{N-1}\setminus N$.

To this end, for each $\xi \in S^{N-1}\setminus N$ consider a subsequence $\{u_k\} \equiv \{u_{h_k}\}$ such that

$$\liminf_{h\to+\infty}\int_{J_{{u_h}^\xi_y}} \psi(|[{u_h}^\xi_y]|(t))d\mathcal{H}^0(t) = \lim_{k\to+\infty}\int_{J_{{u_k}^\xi_y}} \psi(|[{u_k}^\xi_y]|(t))d\mathcal{H}^0(t). \tag{3.5}$$

Next consider a further subsequence (denoted by $\{u_j\} \equiv \{u_{k_j}\}$) such that

$$\lim_{j\to+\infty}\mathcal{H}^{N-1}(J_{u_j}) = \liminf_{k\to+\infty}\mathcal{H}^{N-1}(J_{u_k}). \tag{3.6}$$

We want to show that the assumptions of Lemma 3.1 are satisfied. Let $I_{y,\xi}(u_j) = \int_{\Omega^\xi_y}\theta(|{u'}^\xi_{j\,y}(t)|)dt$, where ${u_j}^\xi_y(t) = u_j(y+t\xi)\cdot\xi$. From (ii) in Theorem 2.1 (i.e. $\mathcal{E}u_j(y+t\xi)\cdot\xi = ({u_j}^\xi)'_y(t)$ for $\mathcal{H}^{N-1}$-a.e. $y \in \Omega^\xi$ and for $\mathcal{L}^1$-a.e. $t \in \Omega^\xi_y$) and from Fubini-Tonelli's theorem, for any $\xi \in S^{N-1}\setminus N$ we have

$$\int_{\pi_\xi} I_{y,\xi}(u_j)d\mathcal{H}^{N-1}(y) = \int_\Omega \theta(|\mathcal{E}u_j(x)\xi\cdot\xi|)dx.$$

Since $\{u_j\}$ satisfies the bound (2.12) and θ is non-decreasing, it follows that

$$\int_{\pi_\xi} I_{y,\xi}(u_j)d\mathcal{H}^{N-1}(y) \le \int_\Omega \theta(|\mathcal{E}u_j(x)|)dx \le K, \tag{3.7}$$

for every $\xi \in S^{N-1}\setminus N$ and for $\mathcal{H}^{N-1}$-a.e. $y \in \Omega^\xi$. It is also easily seen that, from the bound on $\|u_j\|_{L^\infty}$, deriving from the global bound (2.12),

$$\|{u_j}^\xi_y\|_{L^\infty(\Omega^\xi_y)} \le K. \tag{3.8}$$

From Theorem 2.4, (3.7) and (3.8) and Fatou's lemma, for every $\xi \in S^{N-1} \setminus N$ it results that there exists a constant $C \equiv C(K)$ such that

$$\liminf_{j\to+\infty} \int_{\pi_\xi} [I_{y,\xi}(u_j) + \mathcal{H}^0(J_{u_j{}^\xi_y})] d\mathcal{H}^{N-1}(y) \le C < +\infty.$$

Let us fix $\xi \in S^{N-1} \setminus N$ (such that the previous inequality holds). Using Fubini-Tonelli's theorem we can extract a subsequence $\{u_m\} = \{u_{j_m}\}$ (depending on ξ) such that

$$\begin{aligned} &\lim_{m\to+\infty} \int_{\pi_\xi} [I_{y,\xi}(u_m) + \mathcal{H}^0(J_{u_m{}^\xi_y})] d\mathcal{H}^{N-1}(y) = \\ &\liminf_{j\to+\infty} \int_{\pi_\xi} [I_{y,\xi}(u_j) + \mathcal{H}^0(J_{u_j{}^\xi_y})] d\mathcal{H}^{N-1}(y) \le C < +\infty, \end{aligned} \tag{3.9}$$

and for a.e. $y \in \Omega^\xi$, $u^\xi_{m,y} \in SBV(\Omega^\xi_y)$ and $u_m{}^\xi_y \to u^\xi_y$ in $L^1_{loc}(\Omega^\xi_y)$, with $u^\xi_y \in SBV(\Omega^\xi_y)$.

Let $\xi \in S^{N-1} \setminus N$, by (3.9), Fatou's lemma, for $\mathcal{H}^{N-1}$-a.e. $y \in \Omega^\xi$, it results

$$\liminf_{m\to+\infty} [I_{y,\xi}(u_m) + \mathcal{H}^0(J_{u_m{}^\xi_y})] < +\infty. \tag{3.10}$$

Let us fix $N_{\Omega^\xi} \subset \Omega^\xi$ and a point $y \in \Omega^\xi \setminus N_{\Omega^\xi}$, such that $\mathcal{H}^{N-1}(N_{\Omega^\xi}) = 0$, (3.10) and (3.8) hold and such that $u_m{}^\xi_y \in SBV(\Omega^\xi_y)$ for any m. Passing to a further subsequence $\{u_l\} \equiv \{u_{m_l}\}$ we can assume that there exists a constant C' such that

$$\liminf_{m\to+\infty} [I_{y,\xi}(u_m) + \mathcal{H}^0(J_{u_m{}^\xi_y})] = \lim_{l\to+\infty} [I_{y,\xi}(u_l) + \mathcal{H}^0(J_{u_l{}^\xi_y})] \le C'.$$

This means that $\{u_l{}^\xi_y\} \in SBV(\Omega^\xi_y)$ and satisfies all the assumptions of Lemma 3.1 for each interval (connected component) $I \subset \Omega^\xi_y$. Consequently (3.5) and Lemma 3.1 guarantee that

$$\int_{J_{u^\xi_y}} \psi(|[u^\xi_y]|(t)) d\mathcal{H}^0(t) \le \lim_{l\to+\infty} \int_{J_{u_l{}^\xi_y}} \psi(|[u_l{}^\xi_y]|(t)) d\mathcal{H}^0(t) = \liminf_{h\to+\infty} \int_{J_{u_h{}^\xi_y}} \psi(|[u_h{}^\xi_y]|(t)) d\mathcal{H}^0(t) \tag{3.11}$$

for $\mathcal{H}^{N-1}$-a.e. $\xi \in S^{N-1}$ and for $\mathcal{H}^{N-1}$-a.e. $y \in \Omega^\xi$.

The lower semicontinuity stated in (3.4) follows now from Fatou's lemma, which completes the proof. □

In order to prove the main lower semicontinuity theorem with respect to convergence (2.13) ÷ (2.16), we need to apply Lemma 2.8, to the class of functions in (1.4), exploiting also some other properties of the infimal convolutions as in the Lemma below.

Lemma 3.4. *Let $\psi : [0, +\infty[\to [0, +\infty]$ be a lower semicontinuous function. Define for $t > 0$*

$$\psi_t(x) := \inf \{\psi(y) + t|x - y| : y \in [0, +\infty[\}. \tag{3.12}$$

Then

- *(i) if ψ is subadditive, nondecreasing then $\psi_t \le \psi$, ψ_t is continuous, subadditive and nondecreasing and $\psi_t(x) \uparrow \psi(x)$ as $t \uparrow +\infty$.*

Let $\psi : \mathbb{R}^N \to [0, +\infty]$, not identically $+\infty$, and let

$$\psi_t(x) := \inf_{y\in\mathbb{R}^N} \left\{\psi(y) + td(x, y), y \in \mathbb{R}^N\right\}.$$

- *(ii) if ψ depends just on the modulus of x, and the function $x \in \mathbb{R}^N \to \psi(\|x\|)$ is subadditive, then $\psi_t \le \psi$, ψ_t is continuous, the function $x \in \mathbb{R}^N \to \psi_t(\|x\|)$ is subadditive and $\psi_t(x) \uparrow \psi(x)$ as $t \uparrow +\infty$.*

We remark that in (ii) we set, with a notational abuse, $\psi(x) = \psi(\|x\|)$ and similarly for ψ_t.

Proof. The proof is analogous to that of Lemma 2.8, with small modifications. In fact, the continuity of ψ_t, the fact that $0 \leq \psi_t \leq \psi$ and $\psi_t(x) \uparrow \psi(x)$ as $t \uparrow +\infty$ follow from Lemma 2.8. For what concerns the subadditivity, observe that given any function $\Phi : (x,y) \in X \times Y \to \Phi(x,y) \in \mathbb{R}$ (with X and Y closed to addition) which is subadditive in the couple (x,y), then the function $\phi(x) := \inf_{y\in Y} \Phi(x,y)$ is still subadditive in x. Indeed, let $x_1, x_2 \in X$, then $\phi(x_1+x_2) \leq \Phi(x_1+x_2, y_1+y_2) \leq \Phi(x_1,y_1) + \Phi(x_2,y_2)$ for every $y_1, y_2 \in Y$. Since y_1 and y_2 are arbitrary we may pass to the infimum on both terms in the right hand side and obtain

$$\phi(x_1+x_2) \leq \phi(x_1) + \phi(x_2).$$

The above considerations apply to the functions $\Phi(x,y) := \psi(y) + td(x,y)$, d being the Euclidean metric. The other properties may be proved as follows.

(i) Let $(\cdot)^+ : x \in \mathbb{R} \to [0,+\infty[$ be the function defined as

$$(x)^+ := \begin{cases} x & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $(\cdot)^+$ is continuous, nondecreasing and subadditive. The monotonicity of ψ guarantees that (3.12) can be equivalently written as

$$\psi_t(x) = \inf\left\{\psi(y) + t(x-y)^+ : y \in [0,+\infty[\right\} \tag{3.13}$$

Thus the nondecreasing behavior of ψ_t follows by (3.13).

(ii) Finally it easily verified that if $\psi(Ox) = \psi(x)$ for every orthogonal matrix O and for every $x \in \mathbb{R}^N$, then the same invariance is inherited by ψ_t. Consequently if $\psi(\cdot) = \psi(\|\cdot\|)$ and it is subadditive, the same holds for ψ_t.

□

Now we are in position to prove Theorem 1.2.

Proof of Theorem 1.2. (1.2) has been proved in [11] (cf. Lemma 3.1 therein). It remains to prove (1.5). To this end assume first that the function ψ in (1.4) is continuous and $\psi(|\cdot|)$ is subadditive. Indeed, if Ψ is as in (1.4), as observed in Remark 4.3, the continuity of ψ allows us to assume ξ in (1.4) varying in any countable subset of S^{N-1}. It will be chosen in $S^{N-1} \setminus N$, N being the $\mathcal{H}^{N-1}$ exceptional set introduced in Lemma 3.3, and it will be denoted by $\mathcal{A}$, with elements ξ_α. On the other hand, as already observed in Remark 3.2 any function ψ subadditive and nondecreasing is such that $\psi(|\cdot|)$ is subadditive.

By superadditivity of liminf:

$$\liminf_{h\to+\infty} \int_{J_{u_h}} \Psi([u_h], \nu_{u_h}) d\mathcal{H}^{N-1} \geq \sum_\alpha \liminf_{h\to+\infty} \int_{J_{u_h}\cap A_\alpha} |\xi_\alpha \cdot \nu_{u_h}| \psi(|\xi_\alpha \cdot [u_h]|) d\mathcal{H}^{N-1}$$

for any finite family of pairwise disjoint open sets $A_\alpha \subset \Omega$.

By Lemma 3.3 we have

$$\liminf_{h\to+\infty} \int_{J_{u_h}} |\xi_\alpha \cdot \nu_{u_h}| \psi(|\xi_\alpha \cdot [u_h]|) d\mathcal{H}^{N-1} \geq \int_{J_u\cap A_\alpha} |\xi_\alpha \cdot \nu_u| \psi(|\xi_\alpha \cdot [u]|) d\mathcal{H}^{N-1}$$

for every $\xi_\alpha \in \mathcal{A}$. Therefore

$$\liminf_{h\to+\infty} \int_{J_{u_h}} \Psi([u_h], \nu_{u_h}) d\mathcal{H}^{N-1} \geq \sum_\alpha \int_{J_u\cap A_\alpha} |\xi_\alpha \cdot \nu_u| \psi(|\xi_\alpha \cdot [u]|) d\mathcal{H}^{N-1}$$

for every $\xi_\alpha \in \mathcal{A}$ and for any finite family of pairwise disjoint open sets $A_\alpha \subset \Omega$.

By Theorem 2.6 we can interchange integration and supremum over all such families, thus getting

$$\liminf_{h\to+\infty} \int_{J_{u_h}} \Psi([u_h], \nu_{u_h}) d\mathcal{H}^{N-1} \geq \int_{J_u} \Psi([u], \nu_u) d\mathcal{H}^{N-1},$$

whence (1.5) follows.

For what concerns the general case, i.e. ψ lower semicontinuous, (ii) of Lemma 3.4 ensures that any lower semicontinuous function ψ such that $\psi(|\cdot|)$ is subadditive can be seen as the supremum of a countable family of continuous functions ψ_t, such that $\psi_t(|\cdot|)$ is still subadditive.

Let $\{\psi_n\}$ be such a family, i.e. $\psi(t) = \sup_{n\in\mathbb{N}} \psi_n(t)$, for every $t \in [0, +\infty[$, with $\{\psi_n\}$ non decreasing in n. Furthermore, for every $n \in \mathbb{N}$, let $\Psi_n : \mathbb{R}^N \times S^{N-1} \to [0, +\infty[$ be the functional defined by

$$\Psi_n(a, b) := \sup_{\xi\in S^{N-1}} |b \cdot \xi|\psi_n(|a \cdot \xi|). \tag{3.14}$$

Clearly,

$$\Psi(a, b) = \sup_{n\in\mathbb{N}} \Psi_n(a, b). \tag{3.15}$$

Since each supremum is actually a monotone limit, monotone convergence theorem gives

$$\int_{J_u} \Psi([u], \nu_u)d\mathcal{H}^{N-1} = \lim_{n\to+\infty} \int_{J_u} \Psi_n([u], \nu_u)d\mathcal{H}^{N-1}.$$

On the other hand, the first part of the proof ensures that each functional $\int_{J_u} \Psi_n([u], \nu_u)d\mathcal{H}^{N-1}$ is sequentially lower semicontinuous with respect to the L^1- strong convergence along all the sequences $\{u_h\} \subset SBD(\Omega)$ satisfying the bound (2.12), so that

$$\int_{J_u} \Psi([u], \nu_u)d\mathcal{H}^{N-1} \le \liminf_{h\to+\infty} \int_{J_{u_h}} \Psi([u_h], \nu_{u_h})d\mathcal{H}^{N-1},$$

which concludes the proof. □

Remark 3.5. *We observe, as already enlightened in the proof, that Theorem 1.2 holds just by assuming that the function* $\psi : [0, +\infty[\to [0, +\infty[$ *is lower semicontinuous and the function* $x \in \mathbb{R}^N \to \psi(\|x\|)$ *is subadditive. Typical examples of non monotonic functions with this property are* $\psi(t) = |\sin t|$, $(N = 1)$ *and any* ψ *with* $1 \le \psi(t) \le 2, (N \ge 1)$, ψ *continuous, bounded and* $\psi(0) = 0$.

Consequently, as emphasized in Remark 3.2 the results apply to the class of functions $\psi : [0, +\infty[\to [0, +\infty[$*, lower semicontinuous, nondecreasing and subadditive, relevant for the applications to Continuum Mechanics.*

Finally, we emphasize that Theorem 1.2 still holds with obvious adaptations if one replaces the integrand Ψ *in (1.4) by*

$$\Psi(a, b) := \sup_{\xi\in S^{N-1}} |b \cdot \xi|\psi_\xi(|a \cdot \xi|),$$

with $\psi_\xi : [0, +\infty[\to [0, +\infty[$ *continuously depending on* $\xi \in S^{N-1}$*, nondecreasing, subadditive.*

4 Structure properties of the energy densities

This section is devoted to illustrate the properties of the energy density Ψ defined in (1.4).

Proposition 4.1. *Let* Ψ *be defined by formula (1.4) with* $\psi : [0, +\infty[\to [0, +\infty[$*, and such that the function* $t \in [0, +\infty[\to t\psi(t)$ *is nondecreasing in* $[0, +\infty[$*. Then it results*

$$\psi(t) = \Psi(tb, b) \tag{4.1}$$

for every $b \in S^{N-1}$ *and for every* $t \ge 0$.

Proof. Let $b \in S^{N-1}$ and $t \ge 0$. From (1.4) it follows

$$\Psi(tb, b) = \sup_{\xi\in S^{N-1}} |b \cdot \xi|\psi(|tb \cdot \xi|) = \sup_{0\le s\le 1} s\psi(ts) \tag{4.2}$$

It is easily seen that if the function $t \in [0, +\infty[\to t\psi(t)$ is nondecreasing then the right hand side of (4.2) coincides with $\psi(t)$ and that concludes the proof.

□

Clearly Ψ inherits many of the properties of ψ.

Proposition 4.2. *Let Ψ be defined by formula (1.4) with $\psi : [0, +\infty[\to [0, +\infty[$, then the following properties hold.*

(i)

$$\Psi(Oa, Ob) = \Psi(a, b) \tag{4.3}$$

for every orthogonal matrix $O \in \mathbb{R}^{N\times N}$, $a \in \mathbb{R}^N$ and $b \in S^{N-1}$.

(ii) $\Psi(a, \cdot)$ *is an even function, i.e. for every $a \in \mathbb{R}^N$*

$$\Psi(a, b) = \Psi(a, -b)$$

for every $b \in S^{N-1}$ and,

$$\Psi(a, b) = \Psi(-a, b)$$

for every $a \in \mathbb{R}^N$.

(iii) If ψ is lower semicontinuous, then Ψ is lower semicontinuous on $\mathbb{R}^N \times S^{N-1}$,

(iv) If ψ is subadditive, then $\Psi(\cdot, b)$ is subadditive for every $b \in S^{N-1}$.

(v) If ψ is continuous (or Lipschitz), then Ψ is continuous on $\mathbb{R}^N \times S^{N-1}$ (or $\Psi(\cdot, b)$ is Lipschitz for every $b \in S^{N-1}$. The local Lipschitz property of Ψ easily follows).

(vi) If ψ is convex so is $\Psi(\cdot, b)$ for every $b \in S^{N-1}$.

(vii) If ψ is bounded, Ψ is also bounded on $\mathbb{R}^N \times S^{N-1}$.

(viii) If ψ is nondecreasing, then the function $t \in [0, +\infty[\to \Psi(ta, b)$ is nondecreasing for every $a \in \mathbb{R}^N$ and for every $b \in S^{N-1}$.

(ix) $\Psi(a, b) = 0$ *if and only if $\psi(t) = 0$ for every $t \in [0, |a|]$, at least if ψ is continuous.*

(x) $\Psi(a, \cdot)$ *is the restriction to S^{N-1} of a seminorm on $\mathbb{R}^N$.*

In particular, if the function $t \in [0, +\infty[\to t\psi(t)$ is nondecreasing, then Ψ is lower semicontinuous (respectively subadditive, continuous, Lipschitz, convex, bounded, nondecreasing) if and only if ψ shares the corresponding properties.

Proof. For what concerns (i) we observe that

$$\begin{aligned}\Psi(Oa, Ob) &= \sup_{\xi\in S^{N-1}} |Ob\cdot\xi|\psi(|Oa\cdot\xi|) = \sup_{\eta\in S^{N-1}} |Ob\cdot O\eta|\psi(|Oa\cdot O\eta|) = \sup_{\eta\in S^{N-1}} |b\cdot\eta|\psi(|a\cdot\eta|)\\ &= \sup_{\xi\in S^{N-1}} |b\cdot\xi|\psi(|a\cdot\xi|) = \Psi(a, b),\end{aligned}$$

for every orthogonal matrix $O \in \mathbb{R}^{N\times N}$, and for every $(a, b) \in \mathbb{R}^N \times S^{N-1}$.

(ii) is analogous and immediate from (1.4). (iii), (iv), (vi), (vii), (viii) follow from the closure to supremum of the corresponding properties. Concerning (v) we only need to add the further remark that the function $\Psi(\cdot, b)$ is the supremum of uniformly equicontinuous (respectively, uniformly Lispchitz) functions. (ix) follows obviously from (1.4) if ψ is continuous, since the set of ξ where $b \cdot \xi \neq 0$ is dense. The general case (which differs only for a, b parallel or orthogonal) can be treated by proceeding as in Theorem 4.4 below. To prove (x) just observe that $\Psi(a, cdot)$ is the supremum of a family of seminorms restricted to S^{N-1}. Finally, the last statement is a consequence of Proposition 4.1. $\square$

Remark 4.3. *We observe that if ψ is continuous, the supremum in (1.4) can be taken over any dense countable family of S^{N-1}, this property may fail if ψ is not continuous (e.g. upper semicontinuous will not do in general).*

Theorem 4.4. *Let $\psi : [0, +\infty[\to [0, +\infty[$ and let $\Psi : \mathbb{R}^N \times S^{N-1} \to [0, +\infty[$ be as in (1.4). Then there exists a unique function $\tilde{\psi} : [0, +\infty[\to [0, +\infty[$ such that the function $t \in [0, +\infty[\to t\tilde{\psi}(t)$ is nondecreasing and*

$$\Psi(a, b) = \sup_{\xi \in S^{N-1}} |b \cdot \xi| \tilde{\psi}(|a \cdot \xi|). \tag{4.4}$$

Proof. Let Ψ be as in (1.4) and ψ a nonnegative function, then by (ii) of Proposition 4.2 we may further assume that $a \cdot b \geq 0$. Without loss of generality, by (i) of Proposition 4.2, we may assume $a := r\mathbf{e}_1$, $r > 0$, and $b = \cos\gamma \mathbf{e}_1 + \sin\gamma \mathbf{e}_2$, $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$ being the canonical basis in $\mathbb{R}^N$, $\gamma \in \left[0, \frac{\pi}{2}\right]$ the angle between a and b. Consequently, (1.4) becomes

$$\Psi(a, b) = \sup_{\xi \in S^{N-1}} \psi(|r\xi_1|)(\xi_1 \cos\gamma + \xi_2 \sin\gamma),$$

where ξ_1 and ξ_2 are the first two components of ξ. Now, by virtue of the monotonicity in the right hand side in the variables $\xi_2, \cos\gamma$ and $\sin\gamma$, since $\psi \geq 0$, the above maximum problem becomes

$$\sup_{\substack{\xi_1^2 + \xi_2^2 \leq 1 \\ \xi_1, \xi_2 \geq 0}} \psi(r\xi_1)(\xi_1 \cos\gamma + \xi_2 \sin\gamma) =$$

$$\sup_{0 \leq s \leq 1} \sup_{\substack{x^2 + y^2 = 1 \\ x, y \geq 0}} s\psi(rsx)(x \cos\gamma + y \sin\gamma) =$$

$$\sup_{\substack{x^2 + y^2 = 1 \\ x, y \geq 0}} \tilde{\psi}(rx)(x \cos\gamma + y \sin\gamma) =$$

$$\sup_{\substack{\xi_1^2 + \xi_2^2 \leq 1 \\ \xi_1, \xi_2 \geq 0}} \tilde{\psi}(r\xi_1)(\xi_1 \cos\gamma + \xi_2 \sin\gamma)$$

where

$$\tilde{\psi}(u) = \sup_{0 \leq s \leq 1} s\psi(su). \tag{4.5}$$

Thus Ψ can be equivalently obtained by (4.4) and it is easily seen that the function $t \in [0, +\infty[\to t\tilde{\psi}(t)$ is nondecreasing in $[0, +\infty[$.
Finally uniqueness of $\tilde{\psi}$ follows from Proposition 4.1. $\square$

Propositions 4.1, 4.2 and Theorem 4.4 allow us to characterize function Ψ in (1.4), as summarized in the following.

Theorem 4.5. *Let $\Psi : \mathbb{R}^N \times S^{N-1} \to [0, +\infty[$ then*

(E) there exists $\psi : [0, +\infty[\to [0, +\infty[$ such that (1.4) holds if and only if

$$\Psi(a, b) = \sup_{\xi \in S^{N-1}} |b \cdot \xi| \Psi(|a \cdot \xi| e, e) \tag{4.6}$$

for any $e \in S^{N-1}$.

(U) The function ψ in (1.4) is unique among the class of functions such that $(\cdot)\psi(\cdot)$ is nondecreasing.

Proof. (E): (4.6) trivially entails (1.4). The necessary part follows from (4.2) in Proposition 4.1, (4.4) and (4.5) in Theorem 4.4.

(U) has been stated and proved in Theorem 4.4. We also observe that by (4.5) and (4.1), $\tilde{\psi}$ is also the maximum function which recovers representation (1.4). $\square$

Remark 4.6. *We also observe that Theorem 4.5 and Theorem 4.4 provide a complete characterization also in the light of Theorem 1.2. In fact assuming that Ψ is lower semicontinuous, $\Psi(\cdot, b)$ is subadditive for every $b \in \mathbb{R}^N$, and $t \in [0, +\infty[\to \Psi(ta, b)$ is nondecreasing for every $(a, b) \in \mathbb{R}^N \times S^{N-1}$, then (4.5), (4.2), guarantee that the function $\tilde{\psi}$ shares the same properties, i.e. it is lower semicontinuos, subadditive and nondecreasing.*

Examples 4.7. *We show the expression of the function Ψ in (1.4) for some prescribed ψ.*

(i) Consider first the model in [5, 6]: $\psi_{const} : t \in [0, +\infty[\to K$, $K > 0$. It is easily seen that ψ_{const} leads to $\Psi_{const} : (a, b) \in \mathbb{R}^N \times S^{N-1} \to K$.

In the subsequent analysis we may limit, by symmetry ($\Psi(a, b) = \Psi(a, -b)$), our attention to the set $\{(a, b) \in \mathbb{R}^N \times S^{N-1} | a \cdot b \geq 0\}$. Moreover in the Continuum Mechanics setting, a and b play the role of the jump $[u]$ and the normal to the jump ν_u respectively, while the noninterprenetration constraint (1.2) corresponds to the latter set.

(ii) Let $\psi_{lin}(t) = Ct$, for every $t \in [0, +\infty[$, then $\Psi_{lin}(a, b) = \frac{C}{2}(a \cdot b + |a||b|)$, and observe that the maximum value in formula (1.4) is reached on the bisector of the angle between a and b.

(iii) Let ψ_D be the Dugdale function in (3.1), the function Ψ_D in (1.4) turns out to be the following:

$$\Psi_D(a, b) = \begin{cases} \frac{1}{2}|a| (\cos \gamma + 1) & \text{if } |a| \leq \frac{1}{\cos\left(\frac{\gamma}{2}\right)}, \\ \frac{\cos \gamma}{|a|} + \sin \gamma \sqrt{1 - \frac{1}{|a|^2}} & \text{if } \frac{1}{\cos\left(\frac{\gamma}{2}\right)} < |a| \leq \frac{1}{\cos \gamma}, \\ 1 & \text{if } |a| \geq \frac{1}{\cos \gamma} \end{cases}$$

where γ denotes the angle between a and b (i.e. $\cos(\gamma) = \frac{a \cdot b}{|a||b|}$). We also observe that, in the case of the Dugdale function, if the deformations are small ($|a| := |[u]| \leq 1$) the function Ψ_D, obtained from (1.4) is described just by the first line above, i.e. it agrees with example (ii), while in the third regime it agrees with case (i).

(iv) $\psi : t \in [0, +\infty[\to |t|^p$, with $p \in (0, 1)$, then for every a and b, with $b \in S^{N-1}$, $a \cdot b \geq 0$.

$$\Psi(a, b) = |a|^p \left(1 + \frac{\tan^2[\frac{1}{2}\arctan\left(\frac{2\sqrt{p}\tan\gamma}{p+1}\right)]}{p} \right)^{\frac{-p}{2}} \left(1 + p \tan^2 \left[\frac{1}{2}\arctan\left(\frac{2\sqrt{p}\tan\gamma}{p+1}\right)\right]\right)^{\frac{-1}{2}}$$

where γ again denotes the angle between a and b.

It can be easily verified that for $p \nearrow 1$ we recover the example (ii) with $C = 1$. Clearly if $a \perp b$, then $\Psi(a, b) = |a|^p p^{\frac{p}{2}} (1 + p)^{-\frac{p+1}{2}}$. Also if $a \| b$, then $\Psi(a, b) = |a|^p$ in agreement with Proposition 4.1.

Remark 4.8. *The analysis on the properties of the function Ψ in (1.4) leads us to observe that the results contained in Theorem 1.2 are not merely an extension of our previous results on SBD contained in [11]. Indeed in [11] we deal with the lower semicontinuity of $\int_{J_u} \varphi([u] \cdot \nu_u) d\mathcal{H}^{N-1}$, $[u] \cdot \nu_u \geq 0$ $\mathcal{H}^{N-1}$-a.e. on J_u, with $\varphi : [0, +\infty[\to [0, +\infty[$ nondecreasing, convex and subadditive (see (1.1)). We prove that there is lower semicontinuity with respect to convergences (2.13) $\div$ (2.16), by assuming*

$$\varphi(t) = \sup_{\alpha \in \mathcal{A}} \{c_\alpha t + d_\alpha\}, \tag{4.7}$$

with $c_\alpha, d_\alpha \geq 0$. On the other hand, the results contained in Theorem 1.2 concern $\int_{J_u} \Psi([u], \nu_u) d\mathcal{H}^{N-1}$, $[u] \cdot \nu_u \geq 0$ $\mathcal{H}^{N-1}$-a.e. on J_u, with $\Psi : (a, b) \in \mathbb{R}^N \times S^{N-1} \to \sup_{\xi \in S^{N-1}} |b \cdot \xi| \psi(|a \cdot \xi|)$, with ψ nondecreasing, lower semicontinuous, subadditive.
Thus, it is natural to characterize the intersection among the two classes of Theorems 1.1 and 1.2.
Therefore we seek ψ such that

$$\sup_{\xi \in S^{N-1}} |b \cdot \xi| \psi(|a \cdot \xi|) = \varphi(a \cdot b)$$

for every $a \in \mathbb{R}^N$*,* $b \in S^{N-1}$ *and* $a \cdot b \geq 0$*. By (4.1) and Theorem 4.4 we may assume*

$$\psi(t) = \varphi(t), \text{ for every } t \in [0, +\infty[,$$

so that

$$\varphi\left(a \cdot b\right) = \sup_{\xi \in S^{N-1}} |b \cdot \xi| \varphi(|a \cdot \xi|) \text{ for every } (a, b) \in \mathbb{R}^N \times S^{N-1}.$$

which implies

$$\varphi(a \cdot b) \geq \varphi(|a \cdot \xi|)|b \cdot \xi|, \text{ for every } a \in \mathbb{R}^N, b, \xi \in S^{N-1}, a \cdot b \geq 0. \tag{4.8}$$

Taking $0 \leq x, y \leq r$*,* $(t, s) = \frac{1}{r}(x, y)$*,* $a = r\mathbf{e}_1$*,* $b = t\mathbf{e}_1 + \sqrt{1-t^2}\mathbf{e}_2, \xi = s\mathbf{e}_1 + \sqrt{1-s^2}\mathbf{e}_2$*,* $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$ *the canonical basis of* $\mathbb{R}^N$*, and letting* $r \to +\infty$ *we obtain by (4.8)*

$$\varphi(x) \geq \varphi(y), \text{ for every } x, y \geq 0,$$

which ensures that φ *has to be constant.*

By the same token, if Ψ *depends only on* $|a|$*, we see that* $\tilde{\psi}$ *in (4.5) is either a constant or has the form*

$$\tilde{\psi}(t) = \begin{cases} \alpha & t = 0, \\ \beta(> \alpha) & t > 0. \end{cases}$$

and

$$\Psi(a, b) = \begin{cases} \alpha & a = 0, \\ \beta & a \neq 0. \end{cases}$$

Therefore, while the symmetry property (4.3) means that Ψ *depends only on* $a \cdot b$ *and* $|a|$ *(angle between the jump and the normal to the jump and amplitude of the jump), only 'trivial' models for* Ψ *yield a dependence on a single such parameter.*

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