The Theory of Singular Differential-Operator Equations

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Abstract

We consider methods of reduction of differential operator equations with the Fredholm operator in the main expression to regular problems. Relation between the initial conditions choice problem and the Jordan structure of operator coefficients of equations is shown. The theorem of existence and uniqueness of the initial problem is proved. The method of fundamental operators is used for construction of solutions in the Schwarz distribution class.

Key words: Differential operator equations, Jordan sets, Fredholm operator, fundamental operator-functions.

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1 Introduction

Let

$$x = (t, x') \text{ be a point in the space } R^{m+1},$$

$$x' = (x_1, \ldots, x_m), \quad D = (D_t, D_{x_1}, \ldots, D_{x_m}),$$

$$\alpha = (\alpha_0, \ldots, \alpha_m), \mid \alpha \mid = \alpha_0 + \alpha_1 + \cdots \alpha_m, \text{where}$$

$$\alpha_i \text{ are integer non-negative indexes, } D^\alpha = \frac{\partial^\alpha}{\partial t^{\alpha_0} \ldots \partial x_m^{\alpha_m}}.$$

We also suppose that $B_\alpha : D_\alpha \subset E_1 \to E_2$ are closed linear operators with dense domains $D(B_\alpha)$ in E_1, $B \equiv B_{l0\ldots0}$ is a Fredholm operator, $D(B) \subseteq D(B_\alpha) \forall \alpha \; x \in \Omega$, where $\Omega \subset R^{m+1}, \mid t \mid \leq T$, E_1, E_2 are Banach spaces.

We consider the equation

$$L(D)u = f(x), \tag{1}$$

where

$$L(D) = \sum_{|\alpha| \leq l} B_\alpha D^\alpha,$$

$f : \Omega \to E_2$ is an analytical function of x' sufficiently smooth of t.

The Cauchy problem for (**??**), when $E_1 = E_2 = R^n$ and the matrix $B = B_{l0\ldots0}$ is not degenerated, has been thoroughly studied in fundamental papers by I.G. Petrovsky (see [**?**]). In the case when the operator B is not invertible the theory of initial and boundary value problems for (**??**) is not developed even for the case of finite dimensions. In general, the standard Cauchy problem with conditions $D_t^i|_{t=0} u = g_i(x'), \;\; i = 0, \ldots, l-1$ for (**??**) has no classical solutions for an arbitrary right part $f(x)$.

The motif of our investigations is the wish to conceive the statement of initial and boundary value problems for the partial differential systems with the Fredholm operator in the main part and also their applications.

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In this paper we show that we can get a reasonable statement of the initial problems for such systems by decomposing the space E_1 on the direct sum of subspaces in accordance with the Jordan structure ([?],[?], [?]) of the operator coefficients B_α, and imposing conditions on projections of the solution. In this case the projections of the solution will be defined by regular problems.

Here we suppose that B is a closed Fredholm operator, and among the coefficients B_α there is an operator $A = B_{l_1 0 \ldots 0}$, $l_1 < l$, with respect to which B has the complete A-Jordan set [?]. Here P is the projector of E_1 onto corresponding A-root subspace.

In Section 2 the investigation of equation (**??**) is reduced to regular problems and the sufficient conditions of existence of the unique solution of equation (**??**) with the initial conditions

$$D_t^i u|_{t=0} = g_i(x'), \ \ i = 0, 1, \ldots, l_1 - 1, \tag{2}$$

$$(I - P) D_t^i u|_{t=0} = g_i(x'), \ \ i = l_1, \ldots, l - 1, \tag{3}$$

are obtained, where $g_i(x')$ are analytical functions with values in E_1, $Pg_i(x') = 0, \ \ i = l_1, \ldots, l-1$.

In this paper the investigation of equation (**??**) is reduced to regular problems. A method of fundamental operators for construction of the solution in the class of Schwarz distributions [?] is considered in Section 3. These investigations can be useful for solving electrical engineering and some mechanical problems [?],[?], etc.

2 Selection of projection operators and reduction of the initial problem to the Kovalevskaya form

Suppose the following condition is satisfied:

Condition 1. [?] The Fredholm operator B has a complete A- Jordan set ϕ_i^j, B^* has a complete A^*- Jordan set ψ_i^j, $\ i = \overline{1,n}, \ j = \overline{1,p_i}$, and the systems $\gamma_i^{(j)} \equiv A^* \psi_i^{(p_i+1-j)}, z_i^{(j)} \equiv A \phi_i^{(p_i+1-j)}$, where $i = \overline{1,n}, j = \overline{1,p_i}$, corresponding to them, are biorthogonal. Here p_i are the lengths of the Jordan chains of the operator B.

Recall that condition 1 is satisfied if the operator $B + \lambda A$ is continuously invertible when $0 <\mid \lambda \mid < \epsilon$ [?].

We introduce the projectors

$$P = \sum_{i=1}^{n} \sum_{j=1}^{p_i} < ., \gamma_i^{(j)} > \varphi_i^{(j)} \equiv (< ., \Upsilon > \Phi),$$

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{p_i} < ., \psi_i^{(j)} > z_i^{(j)} \equiv (< ., \Psi > Z),$$

generating the direct decompositions

$$E_1 = E_{1k} \oplus E_{1\infty-k}, E_2 = E_{2k} \oplus E_{2\infty-k},$$

where $k = p_1 + \cdots + p_n$ is a root number.

Then any solution of equation **??** can be represented in the form

$$u(x) = \Gamma v(x) + (C(x), \Phi), \tag{4}$$

where $\Gamma = (B + \sum_{i=1}^n < ., \gamma_i^{(1)} > z_i^{(1)})^{-1}$ is a bounded operator [?],

$$v \in E_{2\infty-k}, \ \ C(x) = (C_{11}(x), \ldots, C_{1p_1}(x), \ldots, C_{n1}(x), \ldots, C_{np_n}(x))^T,$$

$$\Phi = (\varphi_1^{(1)}, \ldots, \varphi_1^{(p_1)}, \ldots, \varphi_n^{(1)}, \ldots, \varphi_n^{(p_n)})^T,$$

where T denotes transposition.

The unknown functions $v(x) : \Omega \subset R^{m+1} \to E_{2\infty-k}$ and $C(x) : \Omega \subset R^{m+1} \to R^k$ due to initial conditions (**??**), (**??**), satisfy the following conditions

$$D_t^i v|_{t=0} = \begin{cases} B(I-P) g_i(x'), \ i = 0, \ldots, l_1 - 1, \\ B g_i(x'), \ i = l_1, \ldots, l-1, \end{cases} \tag{5}$$

$$D_t^i C|_{t=0} = \beta_i(x'), \ i = 0, \ldots, l_1 - 1. \tag{6}$$

Here $\beta_i(x')$ are coefficients of projections $Pg_i(x'), \ i = 0, \ldots, l_1 - 1$.

Condition 2. The operator coefficients B_α in (**??**) satisfy at least one of five conditions on $D(B_\alpha)$:

1. $B_\alpha P = QB_\alpha$, i.e. B_α (P,Q)-commute, briefly, $\alpha \in q_0$;
2. $B_\alpha P = 0$, briefly $\alpha \in q_1$;
3. $QB_\alpha = 0$, briefly $\alpha \in q_2$;
4. $(I-Q)B_\alpha = 0$, briefly $\alpha \in q_3$;
5. $B_\alpha(I-P) = 0$, briefly $\alpha \in q_4$;

Theorem 2.1 *Suppose conditions 1 and 2 are satisfied, the function $f(x)$ is an analytical on x' and is a sufficiently smooth on t. Suppose*

1. $(q_2, q_4) \subset q_0$ *or* $(q_1, q_3) \subset q_0$;
2. $QB_\alpha P = 0$ *for all* $\alpha \in (q_0, q_3, q_4) \setminus (l0\ldots0), (l_1 0\ldots0)$.

Then the problem (**??**), (**??**), (**??**) *has the unique classical solution* (**??**).

The proof is carried out by substituting (**??**) into (**??**) and then projecting onto the subspaces $E_{2\infty-k}$ and E_{2k}.

Remark 2.1 *Let the operators B_α in condition 2 depend on x for $\alpha \neq (l0\ldots0), (l_1 0\ldots0)$. If these operator coefficients are analytical on x' and sufficiently smooth on t, then theorem 2.1 is valid. Likewise in [**?**] required smoothness on t for these coefficients and $f(x)$ is defined by maximum length of A-Jordan chains of operator B. If $p = \max(p_1, \ldots, p_n)$, then (see [**?**]) existence of derivatives of the order $p-1$ on t for $f(x)$ and for coefficients B_α is sufficient for validity of theorem 2.1.*

Remark 2.2 *Conditions 1 and 2 in theorem 2.1 can be essentially weakened. For example, if $k = n$, then instead of conditions 1 and 2 of theorem 2.1 we can require:*
α). $\max_{\alpha \in (q_2, q_4)} \mid \alpha \mid < l$;
β). $QB_\alpha P = 0$ *for* $\alpha \in (q_0, q_3, q_4)$, $l_1 < \mid \alpha \mid \leq l$.

3 Fundamental operator-functions of singular differential- operator mappings in Banach spaces

Since the standard Cauchy problem for equation (**??**) with the Fredholm operator $B_{l0\ldots0}$ in general has no classical solution [**?**], it is interesting to extend the notion of a solution and to look for a generalized solution in a distribution space [**?**]. The most interesting is to construct fundamental operator-functions for singular differential operators in Banach spaces which make it possible to obtain generalized solutions in closed forms.

Here we construct the fundamental operator-functions for the following operators

$$B\frac{d}{dt} - A$$

and

$$B\frac{d^2}{dt^2} - A,$$

where B is a Fredholm operator.

3.1 Generalized functions in Banach spaces

Let E be a Banach space and E^* is the conjugate one. Assign to the set $K(E^*)$ of *basic functions* all finite functions of class C^∞ with values in E^*. We denote such functions by $s(t)$. *The support* $supp\, s(t)$ of the basic function $s(t)$ is the closure in R^1 of the set of such points t, for which $s(t) \neq 0$. The basic set $K(E^*)$ is the vector space. This space can be made topological one if we define the convergence in it in the following manner

Definition 3.1 The sequence of functions $s_n(t)$ from $K(E^*)$ converges to the function $s(t) \in K(E^*)$ if:

1. There exists $R > 0$ such that $supp\, s_n(t) \subset [-R;\ R]$ $\forall n \in N$;

2. $\forall\alpha \in N \quad \| s_n^{(\alpha)}(t) - s^{(\alpha)}(t) \| \Longrightarrow 0$ uniformly on $t \in [-R;\ R]$ for $n \to \infty$.

■

The set $K(E^*)$ with introduced convergence is called *the space of basic functions* $K(E^*)$.

Any linear continuous functional from $K(E^*)$ we call *generalized function* . The convergence in the set of generalized functions we define as the weak one. The support, the equality of two generalized functions, the addition and the multiplication on the number for generalized functions are defined in usual manner. Locally integrable by Bochner function $u(t)$ with values in E generates the *regular* generalized function according to the following rule

$$(u(t),\ s(t)) \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} < u(t),\ s(t) > dt,\quad \forall s(t) \in K(E^*).$$

All other generalized functions are called *singular.*

From the set $K'(E)$ of generalized functions we select the special class $K'_+(E)$ of generalized functions the supports of which are bounded from the left by zero. Such functions are, for example, the functions of the form $u(t)g(t)$, where $u(t) \in C^\infty(E),\ g(t) \in \mathcal{D}'_+$ [**?**] or $u(t) \in C^\infty_+(E),\ g(t) \in \mathcal{D}'$, acting by the rule

$$(u(t)g(t),\ s(t)) \stackrel{\text{def}}{=} (g(t),\ < u(t),\ s(t) >),\quad \forall s(t) \in K(E^*).$$

Let E_1, E_2 be Banach spaces, $\mathcal{K}(t) \in \mathcal{L}(E_1; E_2)$ is strongly continuous operator-function of the class C^∞, and also $\mathcal{K}^*(t) \in \mathcal{L}(E_2^*; E_1^*)$ there exists for almost all t, $f(t) \in \mathcal{D}'_+$. Then the formal symbol $\mathcal{K}(t)f(t)$ is called a *generalized operator-function.*

Definition 3.2 The convolution of the generalized operator-function $\mathcal{K}(t)f(t)$ and generalized function $v(t) \in K'_+(E_1)$ we call the generalized function $\mathcal{K}(t)f(t) * v(t) \in K'_+(E_2)$, acting by the formula

$$(\mathcal{K}(t)f(t) * v(t),\ s(t)) \stackrel{\text{def}}{=} (f(t),\ (v(\tau),\ \mathcal{K}^*(t)s(t+\tau))),\quad \forall s(t) \in K(E_2^*).$$

■

In particular, if $v(t) = u(t)g(t)$, where $u(t) \in C^\infty(E_1),\ g(t) \in \mathcal{D}'_+,\ u(t) \in D(\mathcal{K}(\cdot))$, then

$$(\mathcal{K}(t)f(t) * u(t)g(t),\ s(t)) = (f(t),\ (u(\tau)g(\tau),\ \mathcal{K}^*(t)s(t+\tau))) =$$

$$= (f(t),\ (g(\tau),\ < \mathcal{K}(t)u(\tau),\ s(t+\tau) >)).$$

From here we obtain the following equality: $\forall s(t) \in K(E_3^*),\quad A \in \mathcal{L}(E_2, E_3), R(\mathcal{K}(\cdot)) \subset D(A)$

$$A\delta^{(\imath)}(t) * \mathcal{K}(t)f(t) * u(t) = (A\mathcal{K}(t)f(t))^{(\imath)} * u(t),$$

which we take as the definition for the case of closed linear operator A.

Consider the differential operator

$$\mathcal{L}(\frac{d}{dt}) = \frac{d^n}{dt^n} + A_{n-1}\frac{d^{n-1}}{dt^{n-1}} + \ldots + A_1\frac{d}{dt} + A_0,$$

where $A_\imath$ are linear bounded operators from E in E, $\overline{\bigcap_{\imath=1}^n D(A_\imath)} = E$, and corresponding to it generalized operator-function

$$\mathcal{L}(\delta(t)) = I\delta^n(t) + A_{n-1}\delta^{n-1}(t) + \cdots + A_1\delta'(t) + A_0\delta(t).$$

Remark 3.1 *If* $u(t) \in C^n(E)$ *is the solution of the Cauchy problem*

$$\mathcal{L}(\frac{d}{dt})u = f(t),\quad u^{(\imath)}(0) = u_\imath,\quad \imath = 0, 1, \cdots, n-1,$$

where $f(t) \in C(E)$, *then* $u(t)$, *being continued by zero on* $t < 0$, *satisfies in generalized sense [**?**] to convolution equation*

$$\mathcal{L}(\delta(t)) * u(t) = g(t),$$

where

$$g(t) = f(t)\theta(t) + \sum_{i=0}^{n-1} c_i \delta^{(i)}(t).$$

Here

$$c_{n-1} = u_o, \;\; c_{n-2} = A_{n-1}u_0 + u_1, \;\; \ldots, \;\; c_0 = A_1 u_0 + \cdots + A_{n-1}u_{n-2} + u_{n-1},$$

i.a. $g(t) \in K'_+(E)$.

Definition 3.3 The generalized operator-function $\mathcal{E}(t)$ of the order n such that for any $u(t) \in K'_+(E)$ on the basic space $K(E^*)$ the equality

$$\mathcal{L}(\delta(t)) * \mathcal{E}(t) * u(t) = u(t)$$

holds, is called *the fundamental operator-function* of the order n for the operator $\mathcal{L}(\frac{d}{dt})$.

■

Remark 3.2 *Due to the triple convolution property mentioned above this definition makes sense also for the case of closed linear operators A_i.*

Examples 3.1 *The generalized operator-function $\mathcal{E}(t) = e^{At}\theta(t)$ is the fundamental operator-function on the class $K'_+(E)$ for the operator $(\frac{d}{dt} - A)$ with a bounded operator A. Similarly, the generalized operator-function $\mathcal{E}(t) = \frac{\sinh\sqrt{A}t}{\sqrt{A}}\theta(t)$ is fundamental operator-function on the class $K'_+(E)$ for the operator $(\frac{d^2}{dt^2} - A)$, if A is bounded.*

Proposition 3.1 *If $\mathcal{E}(t)$ is the fundamental operator-function of the differential operator $\mathcal{L}(\frac{d}{dt})$ on the class $K'_+(E)$, then for $\forall g(t) \in K'_+(E)$ the generalized function $u(t) = \mathcal{E}(t) * g(t) \in K'_+(E)$ on basic space $K(E^*)$ satisfies convolution equation*

$$\mathcal{L}(\delta(t)) * u(t) = g(t).$$

3.2 Fundamental operator-function of singular differential operators

It is valid the following

Theorem 3.1 *Suppose that condition 1 is satisfied. Then the differential operator $(B\frac{d^2}{dt^2} - A)$ on the class $K'_+(E_2)$ has the fundamental operator-function of the form*

$$\mathcal{E}_2(t) = \Gamma\frac{\sinh(\sqrt{A\Gamma}t)}{\sqrt{A\Gamma}}[I - Q]\theta(t) - \sum_{i=1}^{n}[\sum_{k=0}^{p_i-1}\{\sum_{j=1}^{p_i-k} < \cdot, \psi_i^{(j)} > \varphi_i^{(p_i-k+1-j)}\}\delta^{(2k)}(t)],$$

and the operator $(B\frac{d}{dt} - A)$ on the class $K'_+(E_2)$ has the fundamental operator-function of the form

$$\mathcal{E}_1(t) = \Gamma e^{A\Gamma t}[I - Q]\theta(t) - \sum_{i=1}^{n}[\sum_{k=0}^{p_i-1}\{\sum_{j=1}^{p_i-k} < \cdot, \psi_i^{(j)} > \varphi_i^{(p_i-k+1-j)}\}\delta^{(k)}(t)].$$

As the implication of theorem 3.1 and proposition 3.1 we obtain

Theorem 3.2 *Let the conditions of theorem 3.1 be satisfied and the function $f(t) \in C(t \geq 0)$ accepts values in E_2. Then the Cauchy problem*

$$B\dot{x} = Ax + f(t), \;\; x(0) = x_0$$

has a generalized solution of the class $K'_+(E_1)$ of the form

$$x_1 = \mathcal{E}_1(t) * (f(t)\theta(t) + Bx_0\delta(t)),$$

the Cauchy problem

$$B\ddot{x} = Ax + f(t), \;\; x(0) = x_0, \;\; \dot{x}(0) = x_1$$

has a generalized solution of the class $K'_+(E_1)$ of the form

$$x_2 = \mathcal{E}_2(t) * (f(t)\theta(t) + Bx_1\delta(t) + Bx_0\delta'(t)).$$

Remark 3.3 *Direct computations can show that the generalized solutions $x_1(t)$ and $x_2(t)$ coincide with those constructed in [?] by another method. If additionally we require that the singular components of the generalized solutions $x_1(t)$, $x_2(t)$, are equal to zero, then firstly, generalized solutions $x_1(t)$, $x_2(t)$, coincide with continuous (classical) solutions, and, secondly, these additional conditions define a set of the initial conditions and right-hand sides $f(t)$, for which such problems are solvable in the classes of functions $C^1(t \geq 0)$, $C^2(t \geq 0)$ accordingly.*

Remark 3.4 *If $\overline{R(B)} \neq R(B)$, but A is a bounded operator, then all results of theorems 3.1 and 3.2 are valid if we replace in $\mathcal{E}_1(t)$ $\Gamma e^{A\Gamma t}$ on $e^{\Gamma A t}\Gamma$, in $\mathcal{E}_2(t)$ $\Gamma \frac{\sinh(\sqrt{A\Gamma}t)}{\sqrt{A\Gamma}}$ on $\frac{\sinh(\sqrt{\Gamma A}t)}{\sqrt{\Gamma A}}\Gamma$.*

References

[1] V.Dolezal, *Dynamics of linear systems.* Prague, 1967.

[2] B.V.Loginov, Ju.B.Rusak, *Generalized jordan structure in the problem of the stability of bifurcating solutions.* Nonlinear Analysis, Theory, Methods and Applications, vol.17, 3, (1991), pp.219-232.

[3] F.K.Odgvist, T.Hult, *Krieshfestigkeit metallisther werkstoffe* -Berlin.Springer-Verl.,Gottingen(Heidelberg)1962.

[4] I.Petrovsky, *Uber das Caushysche problem fur system von partiellen Differentialgleichunden.* Math .Sbornic,1937,v.2, N5, pp.815-870.

[5] L.Schwartz, *Theorie des distributions* I,II Paris, 1950–1951.

[6] N.A.Sidorov and E.B. Blagodatskaya, *Differential equations with the Fredholm operator in the leading differential expression.* J. Soviet Math. Dokl. 44(1) (1992) pp. 302–305.

[7] N.A.Sidorov and O.A.Romanova, Differentsial'nye Uravneniya 19 (1983), pp.1516-1526; English transl.in Differential Equations 19 (1983).

[8] N.A.Sidorov and M.V.Falaleev, Differentsial'nye Uravneniya 23 (1987), pp.726-728. (Russian)

[9] N.A.Sidorov, O.A.Romanova *Difference-differential equations with the Fredholm operator in the main part.* Katholicke Universiteit Leuven, Department of computer Science. Belgium. Report TW 243, 1996.

[10] M.M.Vainberg and V.A.Trenogin, *The Theory of Branching of Solutions of Nonlinear Equations*, "Nauka", Moscow, 1969; English trans., Wolters-Noordhoff, Groningen, 1974.

[11] V.S.Vladimirov, *The Equations of Mathematical Physics.*"Nauka", Moscow, 1981, pp. 512.

[12] S.T.Zavalishin, A,N.Sesekin, *Impulse Processes, Models and Applications.* "Nauka", Moscow, 1992.