On application of index technique to the bifurcation problems analysis of nonlinear equations

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Abstract

The existence theorems of bifurcation points of solutions for nonlinear operator equation in Banach spaces are proved. Because of these theorems, the sufficient conditions of bifurcation of solutions of boundary-value problem for Vlasov-Maxwell system are obtained.

Introduction

One[1] of the current problems in natural sciences is study of kinetic Vlasov-Maxwell (VM) system [20] describing a behaviour of many-component plasma. A large literature on the existence of solutions for the VM system is available, for example, see under references [1, 3, 11 etc.] and the references given there. Nevertheless, the problem of bifurcation analysis of VM system, which was first formulated by A.A.Vlasov [20], has appeared very complicated on the background of progress of bifurcation theory in other fields and it remains open up to the present time. There are only some isolated results. In [9, 10] the VM system is reduced to the system of semilinear elliptical equations for special classes of distribution functions introduced in [12]. The relativistic version of VM system for such distributions was considered in [1]. One simple existence theorem of a point of bifurcation is announced in [13], and another one is proved for this system in [14].

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The goal of the present paper is to prove a general existence theorems of bifurcation points of VM system with the given boundary conditions on potentials of an electromagnetic field both the densities of charge and current. Here we apply the results of bifurcation theory from [15, 17, 19] and we use the index theory [2, 7] for the study of bifurcation points of the VM system.

We note, the methods, which have been used in [13, 14] are not sufficient for consideration more general situation, which we study below. In what follows, we consider the many-component plasma consisting of electrons and positively charged ions of various species, which described by the many-particle distribution function $f_i = f_i(r, v)$, $i = 1, \dots, N$. The plasma is confined to a domain $D \subset R^3$ with smooth boundary. The particles are to interact only by self-consistent force fields, collisions among particles being neglected.

The behaviour of plasma is governed by the following version of the stationary VM system [20]

$$v \cdot \partial_r f_i + q_i/m_i(E + \frac{1}{c} v \times B) \cdot \partial_v f_i = 0, \tag{1}$$

$$r \in D \subset R^3, \quad i = 1, \dots, N,$$

$$curl E = 0,$$

$$div B = 0$$

$$div E = 4\pi \sum_{k=1}^{N} q_k \int_{R^3} f_k(r, v) \mathrm{d}v \triangleq \rho, \tag{2}$$

$$curl B = \frac{4\pi}{c} \sum_{k=1}^{N} q_k \int_{R^3} v f_k(r, v) \mathrm{d}v \triangleq j.$$

Here $\rho(r)$, $j(r)$ are the densities of charge and current, and $E(r)$, $B(r)$ are the electrical and the magnetic fields.

We seek the solution E, B, f of VM system (1)-(2) for $r \in D \subset R^3$ with boundary conditions on the potentials and the densities

$$U \mid_{\partial D} = u_{01}, \quad (A, d) \mid_{\partial D} = u_{02}; \tag{3}$$

$$\rho \mid_{\partial D=0}, \quad j \mid_{\partial D} = 0, \tag{4}$$

where $E = -\partial_r U$, $B = curl A$, and U, A be scalar and vector potentials.

We call a solution $E^0,\ B^0,\ f^0$ for which $\rho^0 = 0$ and $j^0 = 0$ in domain D, trivial.

In the present paper we investigate the case of distribution functions of the special form [9]

$$f_i(r,v) = \lambda \hat{f}_i(-\alpha_i v^2 + \varphi_i(r),\ v \cdot d_i + \psi_i(r)) \triangleq \lambda \hat{f}_i(\mathbf{R}, \mathbf{G}) \tag{5}$$

$$\varphi_i : R^3 \to R; \quad \psi_i : R^3 \to R; \quad r \in D \subseteq R^3; \quad v \in R^3;$$

$$\lambda \in R^+; \quad \alpha_i \in R^+ \triangleq [0, \infty); \quad d_i \in R^3,\ \ i = 1, \ldots, N,$$

where functions $\varphi_i,\ \psi_i$, generating the appropriate electromagnetic field $(E,\ B)$, has to be defined.

We are interested in the dependence of unknown functions $\varphi_i,\ \psi_i$ upon parameter λ in distribution (5). Here we study the case, when λ in (5) does not depend on physical parameters α_i and d_i. The general case of a bifurcation problem with $\alpha_i = \alpha_i(\lambda)$, $d_i = d_i(\lambda)$, $\varphi_i = \varphi_i(\lambda, r)$, $\psi_i = \psi_i(\lambda, r)$ will be considered in the following paper.

Definition 1. The point λ^0 is called a bifurcation point of the solution of VM system with conditions (3), (4), if in any neighbourhood of vector $(\lambda^0, E^0, B^0, f^0)$, corresponding to the trivial solution with $\rho^0 = 0$, $j^0 = 0$ in domain D, there is a vector (λ, E, B, f) satisfying to the system (1)-(2) with (3), (4) and for which

$$\| E - E^0 \| + \| B - B^0 \| + \| f - f^0 \| > 0.$$

Let $\varphi_i^0,\ \psi_i^0$ are such constants that the corresponding ρ^0 and j^0, induced by distributions f_i in the medium for $\varphi_i^0,\ \psi_i^0$, are equal to zero in domain D. Then VM system has the trivial solution

$$f_i^0 = \lambda \hat{f}_i(-\alpha_i v^2 + \varphi_i^0,\ v \cdot d_i + \psi_i^0), \quad E^0 = 0, \quad B^0 = \beta d_1, \quad \beta - const \ \text{ for } \forall \lambda.$$

The organization of the present work is as follows:

In section 2 two theorems of existence of bifurcation points for the nonlinear operator equation in Banach space generalizing results on a bifurcation point in [5, 15, 16, 17] are proved. The method of proof of these theorems uses the index theory of vector fields [2, 7] and allows to investigate not only the point, but also the bifurcation surfaces with minimum restrictions on equation.

In section 3 we reduce the problem on a bifurcation point of VM system to the problem on bifurcation point of semilinear elliptic system. Last one is treated as the operator equation in Banach space. We derive the branching equation (BEq) which allows to prove the principal theorem of existence of bifurcation points of VM system because of results of section 2. An essential moment here is that the semilinear system of elliptic equations is potential that reduces to potentiality of BEq.

It follows from obtained results that for the original problem the bifurcation is possible only in the case, when number of species of particles $N \geq 3$.

2. Bifurcation of solutions of nonlinear equations in Banach spaces.

Let $E_1,\ E_2$ are real Banach spaces; Υ be normalized space. Consider the equation

$$Bx = R(x, \varepsilon). \tag{6}$$

Here $B : D \subset E_1 \to E_2$ be closed linear operator with a dense range of definition in E_1. The operator $R(x, \varepsilon)$ with values in E_2 is defined, is continuous and continuously differentiable by Frechet with respect to x in a neighbourhood

$$\Omega = \{x \in E_1,\ \varepsilon \in \Upsilon : \parallel x \parallel < r,\ \parallel \varepsilon \parallel < \varrho\}.$$

Thus, $R(0, \varepsilon) = 0,\ R_x(0, 0) = 0$. Let operator B be Fredholm. Let us introduce the basis $\{\varphi_i\}_1^n$ in a subspace $N(B)$, the basis $\{\psi_i\}_1^n$ in $N(B^*)$, and also the systems $\{\gamma_i\}_1^n \in E_1^*$, $\{z_i\}_1^n \in E_2$ which are biorthogonal to these basises.

Definition 2. The point ε_0 is called a bifurcation point of the equation (6), if in any neighbourhood of point $x = 0,\ \varepsilon_0$ there is a pair (x, ε) with $x \neq 0$ satisfying to the equation (6).

It is well known [19] that the problem on a bifurcation point of (6) is equivalent to the problem on bifurcation point of finite-dimensional system

$$L(\xi, \varepsilon) = 0, \tag{7}$$

where $\xi \in R^n$, $L : R^n \times \Upsilon \to R^n$. We call equation (7) the branching equation (BEq). We wright (6) as the system

$$\tilde{B}x = R(x, \varepsilon) + \sum_{s=1}^{n} \xi_s z_s \tag{8}$$

$$\xi_s =< x, \gamma_s >, \quad s = 1, \dots, n, \tag{9}$$

where $\tilde{B} \overset{def}{=} B + \sum_{s=1}^{n} < \cdot, \gamma_s > z_s$ has inverse bounded. The equation (8) has the unique small solution

$$x = \sum_{s=1}^{n} \xi_s \varphi_s + U(\xi, \varepsilon) \tag{10}$$

at $\xi \to 0$, $\varepsilon \to 0$. Substitution (10) into (9) yields formulas for the coordinates of vector-function $L : R^n \times \Upsilon \to R^n$

$$L_k(\xi, \varepsilon) =< R\Big(\sum_{s=1}^{n} \xi_s \varphi_s + U(\xi, \varepsilon), \varepsilon\Big), \psi_k > . \tag{11}$$

Here derivatives

$$\frac{\partial L_k}{\partial \xi_i} \mid_{\xi=0} =< R_x(0, \varepsilon)(I - \Gamma R_x(0, \varepsilon))^{-1} \varphi_i, \psi_k > \overset{def}{=} a_{ik}(\varepsilon)$$

are continuous in a neighbourhood of point $\varepsilon = 0$, $\| \Gamma R_x(0, \varepsilon) \| < 1$.

Let us introduce a set $\Omega = \{\varepsilon \mid \det[a_{ik}(\varepsilon)] = 0\}$, containing point $\varepsilon = 0$ and the following condition:

A) Suppose that in a neighbourhood of point $\varepsilon_0 \in \Omega$ there is a set S, being Jordan continuum, representable as $S = S_+ \bigcup S_-$, $\varepsilon_0 \in \partial S_+ \bigcap \partial S_-$. Moreover, there is a continuous map $\varepsilon(t)$, $t \in [-1, 1]$ such that $\varepsilon : [-1, 0) \to S_-$, $\varepsilon : (0, 1] \to S_+$, $\varepsilon(0) = \varepsilon_0$, $\det[a_{ik}(\varepsilon(t))]_{i,k=1}^{n} = \alpha(t)$, where $\alpha(t) : [-1, 1] \to R^1$ be continuous function vanishes only at $t = 0$.

Theorem 1. *Assume condition* **A**, *and* $\alpha(t)$ *is monotone increasing function. Then* ε_0 *be a bifurcation point of (6).*

Proof. We take arbitrarily small $r > 0$ and $\delta > 0$. Consider the continuous vector field

$$H(\xi, \Theta) \overset{def}{=} L(\xi, \varepsilon((2\Theta - 1)\delta)) : R^n \times R^1 \to R^n,$$

defined at $\xi, \Theta \in M$, where $M\{\xi, \Theta \mid \| \xi \| = r, \ 0 \le \Theta \le 1\}$.

Case 1. If there is a pair $(\xi^*, \Theta^*) \in M$ for which $H(\xi^*, \Theta^*) = 0$, then by definition 2, ε_0 will be a bifurcation point.

Case 2. We assume that $H(\xi, \Theta) \neq 0$ at $\forall (\xi, \Theta) \in M$ and, hence, ε_0 is not a bifurcation point. Then vector fields $H(\xi, 0)$ and $H(\xi, 1)$ are homotopic on the sphere $\| \xi \| = r$. Consequently, their rotations [6] are coincided

$$J(H(\xi, 0), \| \xi \| = r) = J(H(\xi, 1), (\| \xi \| = r) \tag{12}$$

Since vector fields $H(\xi,0)$, $H(\xi,1)$ and their linearizations

$$L_1^-(\xi) \stackrel{def}{=} \sum_{k=1}^{n} a_{ik}(\varepsilon(-\delta))\xi_k \mid_{i=1}^{n},$$

$$L_1^+(\xi) \stackrel{def}{=} \sum_{k=1}^{n} a_{ik}(\varepsilon(+\delta))\xi_k \mid_{i=1}^{n}$$

are nondegenerated on the sphere $\parallel \xi \parallel = r$, then by smallness of $r > 0$, fields $(H(\xi,0)$, $H(\xi,1)$ are homotopic to the linear parts $L_1^-(\xi)$ and $L_1^+(\xi)$.

Therefore

$$J(H(\xi,0), \parallel \xi \parallel = r) = J(L_1^-(\xi), \parallel \xi \parallel = r) \tag{13}$$

$$J(H(\xi,1), \parallel \xi \parallel = r) = J(L_1^+(\xi), \parallel \xi \parallel = r). \tag{14}$$

Because of nondegeneracy of linear fields $L_1^{\pm}(\xi)$, by the theorem about Kronecker index, the following equalities hold

$$J(L_1^-(\xi), \parallel \xi \parallel = r) sign\alpha(-\delta),$$

$$J(L_1^+(\xi), \parallel \xi \parallel = r) = sign\alpha(+\delta).$$

Since $\alpha(-\delta) < 0$, $\alpha(+\delta) > 0$, then the equality (12) is impossible by (13), (14). Hence, we find a pair $(\xi^*, \Theta^*) \in M$ for which $H(\xi^*, \Theta^*) = 0$ and ε_0 be a bifurcation point.

Remark 1. If the conditions of the theorem 1 are satisfied for $\forall \varepsilon \in \Omega_0 \subset \Omega$, then Ω_0 be a bifurcation set of (6). If moreover, Ω_0 is connected set and its every point is contained in a neighbourhood, which is homeomorphic to some domain of R^n, then Ω_0 is called n- dimensional manifold of bifurcation.

For example, it is true, if $\Upsilon = R^{n+1}$, $n \geq 1$, Ω_0 be a bifurcation set of (6) containing point $\varepsilon = 0$ and $\nabla_\varepsilon \det[a_{ik}(\varepsilon)] \mid_{\varepsilon=0} \neq 0$.

It follows from the theorem 1 at $\Upsilon = R^1$ the generalization [17], and also other known strengthenings of M.A.Krasnoselskii theorem about a bifurcation point of odd multiplicity [6].

Most high results in the theory of bifurcation points were obtained for (6) with potential BEq to ξ, when

$$L(\xi, \varepsilon) = grad_\xi U(\xi, \varepsilon). \tag{15}$$

This condition is valid, if a matrix $[\frac{\partial L_k}{\partial \xi_i}]_{i,k=1}^n$ is symmetric. By differentiation of superposition, one finds from (11) that

$$\frac{\partial L_k}{\partial \xi_i} =< R_x\Big(\sum_{s=1}^n \xi_s\varphi_s + U(\xi,\varepsilon),\varepsilon\Big)\Big(\varphi_i + \frac{\partial U}{\partial \xi_i}\Big), \psi_k >, \tag{16}$$

where according to (8), (10)

$$\varphi_i + \frac{\partial U}{\partial \xi_i} = (I - \Gamma R_x)^{-1}\varphi_i. \tag{17}$$

The operator $I - \Gamma R_x$ is continuously invertible because $\| \Gamma R_x \| < 1$ for sufficiently small by norm ξ and ε. Substituting (17) into (16) we obtain equalities

$$\frac{\partial L_k}{\partial \xi_i} =< R_x(I - \Gamma R_x)^{-1}\varphi_i, \psi_k >,\ i,k = 1,\dots,n.$$

It follows the following claim:

Lemma 1. *In order BEq (7) to be potential it is sufficient that a matrix*

$$\Xi = [< R_x(\Gamma R_x)^m \varphi_i, \psi_k >]_{i,k=1}^n$$

to be symmetric at $\forall(x,\varepsilon)$ *in a neighbourhood of point (0,0).*

Corollary 1. Let all matrices

$$[< R_x(\Gamma R_x)^m \varphi_i, \psi_k >]_{i,k=1}^n,\ \ m = 0,1,2,\dots$$

are symmetric in some neighbourhood of point (0,0). Then BEq (7) be potential.

Corollary 2. Let $E_1 = E_2 = H$, H be Hilbert space. If operator B is symmetric in D, and operator $R_x(x,\varepsilon)$ is symmetric for $\forall(x,\varepsilon)$ in a neighbourhood of point (0,0) in D, then BEq be potential.

In the paper [16] has been done more delicate sufficient conditions of BEq potentiality.

Suppose that BEq (7) is potential. Then it follows from the proof of lemma 1 that the corresponding potential U in (15) has the form

$$U(\xi,\varepsilon) = \frac{1}{2}\sum_{i,k=1}^n a_{i,k}(\varepsilon)\xi_i\xi_k + \omega(\xi,\varepsilon),$$

where $\| \omega(\xi,\varepsilon) \| = 0(| \xi |^2)$ at $\xi \to 0$.

Theorem 2. *Let BEq (7) be potential. Assume condition* **A**). *Moreover, let the symmetrical matrix $[a_{ik}(\varepsilon(t))]$ possesses at least ν_1 positive eigenvalues at $t > 0$ and at least ν_2 positive eigenvalues at $t < 0$, $\nu_1 \neq \nu_2$. Then ε_0 will be a bifurcation point of (6).*

Proof. We take the arbitrary small $\delta > 0$ and we consider the function $U(\xi, \varepsilon((2\Theta-1)\delta))$, defined at $\Theta \in [0,1]$ in a neighbourhood of the critical point $\xi = 0$.

Case 1. If there is $\Theta^* \in [0,1]$ such that $\xi = 0$ is the nonisolated critical point of the function $U(\xi, \varepsilon((2\Theta^*-1)\delta)$, then by definition 2, ε_0 will be a bifurcation point.

Case 2. Assume that point $\xi = 0$ will be the isolated critical point of the function $U(\xi, \varepsilon((2\Theta-1)\delta))$ at $\forall\Theta \in [0,1]$, where $\varepsilon(t)$ be continuous function from condition **A**). Then at $\forall\Theta \in [0,1]$, the Conley index [2] K_Θ of the critical point $\xi = 0$ of this function is defined. Let us remind that

$$\det \left\| \frac{\partial^2 U(\xi, \varepsilon((2\Theta-1)\delta))}{\partial\xi_i \partial\xi_k} \right\|_{\xi=0} = \alpha((2\Theta-1)\delta).$$

Since $\alpha((2\Theta-1)\delta) \neq 0$ at $\Theta \neq \frac{1}{2}$, then the critical point $\xi = 0$ at $\Theta \neq \frac{1}{2}$ is nonsingular. Therefore, index K_Θ for any $\Theta \neq \frac{1}{2}$ by the definition (p.6 [2]), is necessary equal to number of positive eigenvalues of the corresponding Hessian. Thus, $K_\Theta = \nu_1$, $K_1 = \nu_2$, where $\nu_1 \neq \nu_2$ by the condition of theorem 2. Hence, $K_\Theta \neq K_1$. Suppose that ε_0 is not a bifurcation point. Then $\nabla_\xi U(\xi, \varepsilon((2\Theta-1)\sigma) \neq 0$ at $0 < \| \xi \| \leq r$, where $r > 0$ is small enough, $\Theta \in [0,1]$. Because of homotopic invariancy of Conley index (see theorem 4, p.52 in [2]), K_Θ is constant at $\Theta \in [0,1]$ and $K_0 = K_1$. Hence, in the second case we find a pair (ξ^*, Θ^*) for arbitrary small $r > 0$, $\delta > 0$, where $0 < \| \xi^* \| \leq r$, $\Theta^* \in [0,1]$, satisfying to the equation $\nabla_\xi U(\xi, \varepsilon((2\Theta-1)\delta) = 0$ and ε_0 is a bifurcation point.

Remark 2. Other proof of the theorem 2 with application of the Roll theorem is given in [18] for the case $\Upsilon = R^1$, $\nu_+ = n$, $\nu_- = 0$.

Remark 3. The theorems 1, 2 (see remark 1) allow to construct not only the bifurcation points, but also the bifurcation sets, surfaces and curves of bifurcation.

Corollary 3. Let $\Upsilon = R^1$ and BEq be potential. Moreover, let $[a_{ik}(\varepsilon)]_{i,k=1}^n$ be positively definite matrix at $\varepsilon \in (0,r)$ and negatively defined at $\varepsilon \in (-r,0)$. Then $\varepsilon = 0$ is a bifurcation point of (6).

Consider the connection of eigenvalues of matrix $[a_{ik}(\varepsilon)]$ with eigenvalues of operator $B - R_x(0, \varepsilon)$.

Lemma 2. *Let $E_1 = E_2 = E$, $\varepsilon \in R^1$; $\nu = 0$ be isolated Fredholm point of operator-function $B - \nu I$. Then*

$$sign \triangle(\varepsilon) = (-1)^k sign \prod_i^k \nu_i(\varepsilon) = sign \prod_i^n \mu_i(\varepsilon),$$

where k be a root number of operator B; $\{\mu\}_1^n$ are eigenvalues of matrix $[a_{ik}(\varepsilon)]$, $\triangle(\varepsilon) = \det[a_{ik}(\varepsilon)]$.

Proof. Since $\{\mu_i\}_1^n$ are eigenvalues of matrix $[a_{ik}(\varepsilon)]$, then $\prod_i^n \mu_i(\varepsilon) = \triangle(\varepsilon)$. Thus, it is sufficient to prove the equality $\triangle(\varepsilon) = (-1)^k \prod_i^k \nu_i(\varepsilon)$. Since zero is the isolated Fredholm point of operator-function $B - \nu I$, then operators B and B^* have the corresponding complete Jordan systems [19]

$$\varphi_i^{(s)} = (\Gamma)^{s-1}\varphi_i^{(1)}, \;\; \psi_i^{(s)} = (\Gamma^*)^{s-1}\psi_i^{(1)}, \;\; i = 1, \dots, n; \;\; s = 1, \dots, P_i. \quad (18)$$

Here

$$< \varphi_i^{(P_i)}, \psi_j >= \delta_{ij}; \;\; < \varphi_i, \psi_j^{(P_j)} >= \delta_{ij}, \;\; i, j = 1, \dots, n; \; \sum_{i=1}^n P_i = k.$$

Let us remind that

$$\varphi_i^{(1)} \triangleq \varphi_i = \Gamma\varphi_i^{(P_i)}, \;\; \psi_i^{(1)} \triangleq \psi_i = \Gamma^*\psi_i^{(P_i)}, \;\; \Gamma = \left(B + \sum_1^n < \cdot, \psi_i^{(P_i)} > \varphi_i^{(P_i)}\right)^{-1}, \quad (19)$$

where $k = l_1 + \ldots + l_n$ we call a root number of operator $B - R_x(0, \varepsilon)$. The small eigenvalues $\nu(\varepsilon)$ of operator $B - R_x(0, \varepsilon)$ satisfy to the following branching equation [19]

$$L(\nu, \varepsilon) \triangleq \det |< R_x(0, \varepsilon) + \nu I)(I - \Gamma R_x(0, \varepsilon) - \nu\Gamma)^{-1}\varphi_i, \psi_j >|_{i,j=1}^n = 0. \quad (20)$$

Because of preliminary Weierstrass theorem [19, p.66], by the equalities (18), (19), equation (20) in a neighbourhood of zero will be transformed to the form

$$L(\nu, \varepsilon) \equiv (\nu^k + H_{k-1}(\varepsilon)\nu^{k-1} + \ldots + H_0(\varepsilon))\Omega(\varepsilon, \nu) = 0,$$

where $H_{k-1}(\varepsilon), \ldots, H_0(\varepsilon) = \triangle(\varepsilon)$ are continuous functions of ε, $\Omega(0, 0) \neq 0$, $H_0(0) = 0$. Consequently, operator $B - R_x(0, \varepsilon)$ has $k \geq n$ small eigenvalues $\nu_i(\varepsilon)$, $i = 1, \ldots, n$, which we may define from the equation

$$\nu^k + H_{k-1}(\varepsilon)\nu^{k-1} + \ldots + \triangle(\varepsilon) = 0.$$

Then $\prod_i^k \nu_i(\varepsilon) = \triangle(\varepsilon)(-1)^k$.

Assume now $\varepsilon \in R^1$. Consider the calculation of asymptotics of eigenvalues $\mu(\varepsilon)$ and $\nu(\varepsilon)$. Let us introduce the block representation of matrix $[a_{ik}]_{i,k=1}^n$, satisfying the following condition:

B) Let $[a_{ik}(\varepsilon)]_{i,k=1}^n = [A_{ik}(\varepsilon)]_{i,k=1}^l \sim [\varepsilon^{r_{ik}} A_{ik}^0]_{i,k=1}^l$ at $\varepsilon \to 0$, where $[A_{ik}]$ are blocks of dimensionality $[n_i \times n_k]$, $n_1 + \ldots + n_l = n$, $min(r_{i1}, \ldots, r_{il}) = r_{ii} \stackrel{\triangle}{=} r_i$ $r_{ik} > r_i$ at $k > i$ (or at $k < i$), $i = 1, \ldots, l$. Let $\prod_1^l \det[A_{ii}^0] \neq 0$. The condition **B)** means that matrix $[a_{ik}(\varepsilon)]_{i,k=1}^n$ admits the block representation being "asymptotically trianglar" at $\varepsilon \to 0$.

Lemma 3. *Assume* **B)**. *Then*

$$\det[a_{ik}(\varepsilon)]_{i,k=1}^n = \varepsilon^{n_1 r_1 + \ldots + n_l r_l}\Big(\prod_1^l \det \mid A_{ii}^0 \mid + 0(1)\Big),$$

formulas

$$\mu_i = \varepsilon^{r_i}(\mathbf{C}_i + 0(1)), \ \ i = 1, \ldots, l \tag{21}$$

define the principal terms of all n eigenvalues of matrix $\mid a_{ik}(\varepsilon) \mid_{i,k=1}^n$, *where* μ_i, $\mathbf{C}_i \in R^{n_i}$; $\mathbf{C}_i$ *be vector of eigenvalues of matrix* A_{ii}^0.

Proof. By **B)** and the property of linearity of determinant, we have

$$\det[a_{ik}(\varepsilon)] = \varepsilon^{n_1 r_1 + \ldots + n_l r_l} \det \begin{vmatrix} A_{11}^0 + 0(1), & 0(1) \ldots\ldots, & \ldots\ldots 0(1) \\ A_{21}^0 + 0(1), & A_{22}^0 + 0(1), & 0(1) \ldots 0(1) \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots \\ A_{l1}^0 + 0(1), & \ldots\ldots, & A_{ll}^0 + 0(1) \end{vmatrix} =$$

$$\varepsilon^{n_1 r_1 + \ldots + n_l r_l}\Big(\prod_i^l \det \mid A_{ii}^0 \mid + 0(1)\Big).$$

Substituting $\mu = \varepsilon^{r_i} c(\varepsilon)$, $i = 1, \ldots, l$ into equation $\det \mid a_{ik}(\varepsilon) - \mu\delta_{ik} \mid_{i,k=1}^n = 0$ and using the property of linearity of determinant we obtain equation

$$\varepsilon^{n_1 r_1 + \ldots + n_{i-1} r_{i-1} + (n_i + \ldots + n_l) r_i}\{\prod_{j=1}^{i-1} \det \mid A_{jj}^0 \mid \cdot$$

$$\det(A_{ii}^0 - (\varepsilon)E)(\varepsilon)^{n_{i+1} + \ldots + n_l} + a_i(\varepsilon)\} = 0, \ \ i = 1, \ldots, l, \tag{22}$$

where $a_i(\varepsilon) \to 0$ at $\varepsilon \to 0$. Hence, the coordinates of unknown principal terms $\mathbf{C}_i$ in asymptoticses (21) satisfy to the equations $\det \mid A_{ii}^0 - E \mid = 0$, $i = 1, \ldots, l$.

If $k = n$, then operator $B - R_x(0, \varepsilon)$, as well as the matrix $[a_{ik}(\varepsilon)]_{i,k=1}^n$ has n small eigenvalues. In this case we state a result:

Corollary 4. Let operator B has not I- joined elements and let the condition **B)** holds. Then the formula

$$\nu_i = -\varepsilon^{r_i}(\mathbf{C}_i + 0(1)), \quad i = 1, \ldots, l, \tag{23}$$

defines all n small eigenvalues of operator $B - R_x(0, \varepsilon)$, where $\mathbf{C}_i \in R^{n_i}$ be vector of eigenvalues of matrix A_{ii}^0, $i = 1, \ldots, l$, $n_1 + \ldots + n_l = n$.

Proof. By lemma 2 in this case $\sum_1^n P_i = n$ (root number $k = n$) and operator $B - R_x(0, \varepsilon)$ possesses at least n small eigenvalues. Since $\sum_1^l n_i = n$, A_{ii}^0 is quadratic matrix, then formula (23) yields n eigenvalues, where the principal terms coincide to within a sign with principal terms in (21). For calculation of eigenvalues ν of operator $B - R_x(0, \varepsilon)$ we transform (20) to the form

$$L(\nu, \varepsilon) \equiv \det[a_{ik}(\varepsilon) + \sum_{j=1}^{\infty} b_{ik}^{(j)} \nu^j]_{i,k=1}^n = 0, \tag{24}$$

where

$$b_{ik}^{(j)} =< [(I - \Gamma R_x(0, \varepsilon))^{-1}\Gamma]^{j-1}(I - \Gamma R_x(0, \varepsilon))^{-1}\varphi_i, \gamma_k > .$$

Substituting $\nu = -\varepsilon^{r_i} c(\varepsilon)$ into (24) and taking into account the property of linearity of determinant we shall receive the equation, which differs from (22) by error term $a_i(\varepsilon)$ only. Then in conditions of corollary 4 the principal terms of all small eigenvalues of operator $B - R_x(0, \varepsilon)$ and matrix $-[a_{ik}(\varepsilon)]$ are defined from the same equations and therefore, are equal.

Conclusions. 1) By lemma 3 we can replace condition **A)** in the theorem 1 with the following one:

A*). Let $E_1 = E_2 = E$; $\nu = 0$ be isolated Fredholm point of operator-function $B - \nu I$. Let in a neighbourhood of point $\varepsilon_0 \in \Omega$ there is a set S, containing point ε_0 and be continuum represented as $S = S_+ \bigcup S_-$. Moreover, assume

$$\varepsilon_0 \in \partial S_+ \bigcap \partial S_-; \quad \prod_i \nu_i(\varepsilon) \mid_{\varepsilon \in S_+} \cdot \prod_i \nu_i(\varepsilon) \mid_{\varepsilon \in S_-} < 0,$$

where $\{\nu_i(\varepsilon)\}$ are small eigenvalues of operator $B - R_x(0, \varepsilon)$.

2) If the principal terms of asymptotics of small eigenvalues of operator $B - R_x(0,\varepsilon)$ and matrix $[a_{ik}(\varepsilon)]_{i,k=1}^n$ coincide, then we may use eigenvalues of such operator in the theorem 2. By corollary 4 it is possible, if $E_1 = E_2 = H$, operators B and $R_x(0,\varepsilon)$ are symmetric and condition B) is valid. Let us note that condition B) is valid in papers [15-18] about bifurcation point with potential BEq, thus $r_1 = \ldots = r_n = 1$.

3. Statement of boundary-value problem and problem on a bifurcation point for the system (32).

We begin with a one preliminary result on reduction of VM system (1)-(2) with conditions (3) to the quasilinear system of elliptical equations for distribution (5), was first investigated in [10]. Assume the following condition:

C). $\hat{f}_i(\mathbf{R},\mathbf{G})$ are fixed, differentiable functions in distribution (5); α_i, d_i are free parameters; $\mid d_i \mid \neq 0$; $\varphi_i = c_{1i} + l_i\varphi(r)$, $\psi_i = c_{2i} + k_i\psi(r)$; c_{1i}, c_{2i}-const; the parameters l_i, k_i are connected by relations

$$l_i = \frac{m_1}{\alpha_1 q_1}\frac{\alpha_i q_i}{m_i}, \quad k_i\frac{q_1}{m_1}d_1 = \frac{q_i}{m_i}d_i, \quad k_1 = l_1 = 1, \tag{25}$$

and the integrals $\int_{R^3}\hat{f}_i \mathrm{d}v$, $\int_{R^3}\hat{f}_i v \mathrm{d}v$ converge at $\forall\varphi_i$, ψ_i.

Let us introduce notations $m_1 \triangleq m$, $\alpha_1 \triangleq \alpha$, $q_1 \triangleq q$.

Theorem 3. *Let f_i are defined as well as in (5) and the condition* **C**) *is valid. Let the vector-function (φ,ψ) is a solution of the system of equations*

$$\triangle\varphi = \mu\sum_{k=1}^N q_k\int_{R^3} f_k \mathrm{d}v, \quad \mu = \frac{8\pi\alpha q}{m}$$

$$\tag{26}$$

$$\triangle\psi = \nu\sum_{k=1}^N q_k\int_{R^3}(v,d) f_k \mathrm{d}v, \quad \nu = -\frac{4\pi q}{mc^2}$$

$$\varphi \mid_{\partial D} = -\frac{2\alpha q}{m}u_{01}, \quad \psi \mid_{\partial D} = \frac{q}{mc}u_{02} \tag{27}$$

on a subspace

$$(\partial_r\varphi_i, d_i) = 0, \quad (\partial_r\psi_i, d_i) = 0, \quad i = 1,\ldots,N. \tag{28}$$

Then the VM system (1), (2) with conditions (5) possesses a solution

$$E = \frac{m}{2\alpha q}\partial_r\varphi, \quad B = \frac{d}{d^2}(\beta + \int_0^1 (d\times J(tr), r)\mathrm{d}t) - [d\times\partial_r\psi]\frac{mc}{qd^2} \qquad (29),$$

where
$$J \triangleq \frac{4\pi}{c}\sum_{k=1}^{N} q_k \int_{R^3} v f_k \mathrm{d}v, \quad \beta - const.$$
The potentials
$$U = -\frac{m}{2\alpha q}\varphi, \quad A = \frac{mc}{qd^2}\psi d + A_1(r), \quad (A_1, d) = 0 \tag{30}$$
satisfying to condition (3) are defined through this solution.

The proof of theorem 3 follows from theorem 1 of paper [14].

Introduce notations
$$j_i = \int_{R^3} v f_i \mathrm{d}v, \quad \rho_i = \int_{R^3} f_i \mathrm{d}v, \quad i = 1, \ldots, N$$
and the following condition:

D). There are vectors $\beta_i \in R^3$ such that $j_i = \beta_i \rho_i$, $i = 1, \ldots, N$.

For example, the condition **D** holds for distribution
$$f_i = f_i(a(-\alpha_i v^2 + \varphi_i) + b((d_i, v) + \psi_i)) \tag{31}$$
for $\beta_i = \frac{b}{2\alpha_i a} d_i$, a, b-const.

Suppose that condition **D** is valid. Then the system (26) will be transformed to the following
$$\triangle\varphi = \lambda\mu \sum_{i=1}^{N} q_i A_i, \quad \triangle\psi = \lambda\nu \sum_{i=1}^{N} q_i (\beta_i, d) A_i, \tag{32}$$
where
$$A_i(l_i\varphi, k_i\psi, \alpha_i, d_i) \triangleq \int_{R^3} \hat{f}_i \mathrm{d}v.$$

Further, we shall suppose that the auxiliary vector d in (5) is directed along axes Z. Because of conditions (28) we put in system (32) $\varphi = \varphi(x, y)$, $\psi = \psi(x, y)$, $x, y \in D \subset R^2$. Moreover, let $N \geq 3$ and $\frac{k_i}{l_i} \neq const$.

Let D be bounded domain in R^2 with the boundary ∂D of class $C^{2,\alpha}$, $\alpha \in (0, 1]$. The boundary conditions (4) on the densities of local charge and current induce the equalities:

I.
$$\sum_{k=1}^{N} q_k A_k(l_k\varphi^0, k_k\psi^0, \alpha_i, d_i) = 0; \quad \sum_{k=1}^{N} q_k(\beta_k, d) A_k(l_k\varphi^0, k_k\psi^0, \alpha_i, d_i) = 0 \tag{33}$$

for $\forall\varepsilon \in \iota$, where ι is a neighbourhood of point $\varepsilon = 0$ and

$$\varphi^0 = -\frac{2\alpha q}{m} u_{01}, \quad \psi^0 = \frac{q}{mc} u_{02}. \tag{34}$$

Remark 4. If $N = 2$ and $\beta_i = \frac{d_i}{2\alpha_i}$, then by condition **I** and equalities $(\beta_i, d) = \frac{d^2}{2\alpha}\frac{k_i}{l_i}$ we have alternative: or in condition **I**: $A_1 = A_2 = 0$ or $k_i = l_i$, $i = 1, 2$. In this case, and also at $\frac{k_i}{l_i}$=const the system (32) is reduced to one equation and bifurcation of solutions in such approach, as it is considered in this paper is impossible.

By (33), (34) system (32) with boundary conditions

$$\varphi \mid_{\partial D} = \varphi^0, \quad \psi \mid_{\partial D} = \psi^0 \tag{35}$$

has a trivial solution $\varphi = \varphi^0$, $\psi = \psi^0$ at $\forall\lambda \in R^+$.

Then because of theorem 3 the VM system with boundary conditions (3), (4) has a trivial solution at $\forall\lambda$

$$E^0 = \frac{m}{2\alpha q}\partial_r \varphi^0 = 0, \quad B^0 = \beta d_1, \quad r \in D \subset R^2,$$

$$f^0 = \lambda \hat{f}_i(-\alpha_i v^2 + c_{1i} + l_i\varphi^0, \ (v, d_i) + c_{2i} + k_i\psi^0).$$

Thus, the densities ρ and j vanish at domain D.

Now our purpose is to find λ_0 in neighbourhood of which system (32), (35) has a nontrivial solution. Then the corresponding densities ρ and j will be identically vanish at domain D, and the point λ_0 is a bifurcation point of the VM system with conditions (4), (5).

Let functions f_i are analytical in (5). Using the expansion in Taylor series

$$A(x, y) = \sum_{i\geq 0}^{\infty} \frac{1}{i!}((x - x^0)\frac{\partial}{\partial x} + (y - y^0)\frac{\partial}{\partial y})^i A(x^0, y^0)$$

and selecting linear terms, we transform (32) to operator form

$$(L_0 - \lambda L_1)u - \lambda r(u) = 0. \tag{36}$$

Here

$$L_0 = \begin{bmatrix} \triangle & 0 \\ 0 & \triangle \end{bmatrix}, \quad u = (\varphi - \varphi^0, \ \psi - \psi^0)'; \tag{37}$$

$$L_1 = \sum_{s=1}^{N} q_s \begin{bmatrix} \mu l_s \frac{\partial A_s}{\partial x} & \mu k_s \frac{\partial A_s}{\partial y} \\ \nu l_s(\beta_s, d) \frac{\partial A_s}{\partial x} & \nu k_s(\beta_s, d) \frac{\partial A_s}{\partial y} \end{bmatrix}_{x=l_s\varphi^0, y=k_s\psi^0} \triangleq$$

$$\begin{bmatrix} \mu T_1 & \mu T_2 \\ \nu T_3 & \nu T_4 \end{bmatrix}; \tag{38}$$

$$r(u) = \sum_{i\geq l}^{\infty} \sum_{s=1}^{n} \varrho_{is}(u) b_s, \tag{39}$$

where

$$\varrho_{is}(u) \triangleq \frac{q_s}{i!}(L_s u_1 \frac{\partial}{\partial x} + k_s u_2 \frac{\partial}{\partial y})^i A_s(l_s\varphi^0, k_s\psi^0)$$

are i homogeneous forms by u;

$$\frac{\partial^{i_1+i_2}}{\partial x^{i_1} \partial y^{i_2}} A_s(x,y) \mid_{x=l_s\varphi^0, y=k_s\psi^0} = 0 \ \text{ at}$$

$$2 \leq i_1 + i_2 \leq l-1, \ \ s = 1, \ldots, N; \ \ l \geq 2; \ \ b_s \triangleq (\mu, \nu(\beta_s, d))'.$$

We study the problem of existence of a bifurcation point λ^0 for (32), (34) as the problem on bifurcation point for operator equation (36).

Let us introduce Banach spaces $C^{2,\alpha}(\bar{D})$ and $C^{0,\alpha}(\bar{D})$ with norms $\| \cdot \|_{2,\alpha}$, $\| \cdot \|_{0,\alpha}$ and $W^{2,2}(D)$, which is usual L^2 Sobolev space in D.

Let us introduce Banach space E of vectors $u \triangleq (u_1, u_2)'$, where $u_i \in L_2(D)$, L_2 be real Hilbert space with internal product (,) and the corresponding norm $\| \cdot \|_{L_2}$ (D). As a range of definition $D(L_0)$ we take set of vectors $u \triangleq (u_1, u_2)$ with $u_i \in \overset{\circ}{W}{}^{2,2}(D)$. Here $\overset{\circ}{W}{}^{2,2}(D)$ denotes $W^{2,2}$ functions with trace 0 on ∂D. Hence, $L_0 : D \subset E \to E$ is linear self-adjoint operator. By virtue of embedding

$$W^{2,2}(D) \subset C^{0,\alpha}(\bar{D}), \ \ 0 < \alpha < 1 \tag{40}$$

the operator $r : W^{2,2} \subset E \to E$ be analytical in neighbourhood of zero. The operator $L_1 \in L(E \to E)$ is linear bounded. For matrix corresponding to operator L_1 we shall keep same notations. By embedding (40) any solution of the equation (36) will be Hölder in $D(L_0)$. Moreover, because the coefficients of (36) are constant, then vector $r(u)$ will be analytical, $\partial D \in C^{2,\alpha}$ and thanks to well-known results of the regularity theory of weak solutions [8], the being searching generalized solutions of (36) in $\overset{\circ}{W}^{2,2}(D)$ belong to $C^{2,\alpha}$.

By theorem 3 on reduction of VM system, the bifurcation points of problem (32), (34) are the bifurcation points of solutions of VM system (1), (2) with boundary conditions (3), (4).

Thanks to given conditions on L_0 and L_1, all singular points of operator-function $L(\lambda) \stackrel{\triangle}{=} L_0 - \lambda L_1$ be Fredholm.

The bifurcation points of nonlinear equation (36) we can found only among points of a spectrum for linearized system

$$(L_0 - \lambda L_1)u = 0. \tag{41}$$

For study of spectrum problem (41) we preliminary find the eigenvalues and the eigenfunctions of matrix L_1 in (41) for physically admissible parameters. With this purpose, we introduce the following condition:

II: $(T_1T_4 - T_2T_3) > 0,\ T_1 < 0.$

Lemma 4. *Let* $\frac{\partial A_i}{\partial x} = \frac{\partial A_i}{\partial y} > 0$, $i = 1, \ldots, N$ *at* $x = l_i\varphi^0$, $y = k_i\psi^0$. *Assume*

$$\sum_{i=2}^{N}\sum_{j=1}^{i-1} a_i a_j (l_j k_i - k_j l_i)(\beta_i - \beta_j, d) > 0,$$

where $a_i \stackrel{\triangle}{=} q_i \frac{\partial A_i}{\partial x}$, *then condition* **II** *is valid.*

Proof. Without loss of generality we put $q \stackrel{\triangle}{=} q_1 < 0$, $q_i > 0$, $i = 2, \ldots, N$. Then via (25) $sign q_i l_i = sign q$. Further, because of definition of T_1 (see.(38)), we verify that $T_1 < 0$. The positiveness of $T_1T_4 - T_2T_3$ follows from equality

$$T_1T_4 - T_2T_3 = \sum l_i a_i \sum k_i(\beta_i, d)a_i - \sum k_i a_i \sum l_i(\beta_i, d)a_i =$$

$$\sum_{i=2}^{N}\sum_{j=1}^{i-1} a_i a_j (l_j k_i - k_j l_i)(\beta_i - \beta_j, d).$$

Example. If $\beta_i = \frac{d_i}{2\alpha_i}$, then $(\beta_i, d) = \frac{d^2}{2\alpha}\frac{k_i}{l_i}$ and

$$\sum_{i=2}^{N}\sum_{j=1}^{i-1} = a_i a_j (l_j k_i - l_i k_j)^2 \cdot \frac{d^2}{2\alpha l_i l_j} > 0.$$

Lemma 5. *Let distribution function has a form (31) and* $f_i' > 0$. *Then conditions* **D** *and* **II** *hold for* $\beta_i = \frac{b}{a}\frac{d_i}{2\alpha_i}$, *and the system (32) will be transformed to the potential form*

$$\triangle \begin{bmatrix} \varphi \\ \\ \psi \end{bmatrix} = \lambda \begin{bmatrix} a_1 & 0 \\ \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} \frac{\partial V}{\partial \varphi} \\ \\ \frac{\partial V}{\partial \psi} \end{bmatrix}, \tag{42}$$

where

$$V = \sum_{k=1}^{N} \frac{q_k}{l_k} \int_0^{al_k\varphi + bk_k\psi} A_k(s)ds, \quad a_1 = \mu/a, \quad a_2 = \frac{\nu d^2}{2ab}. \tag{43}$$

The proof is conducted by direct substitution (43) into the system (42).

Lemma 6. *Let* $r \triangleq x \in R^1$, $v \in R^2$, $d \triangleq d_2$. *Then the system (32) with potential (43) can be written as Hamiltonian system*

$$\dot{p}_\varphi = -\partial_\varphi H, \quad \dot{\varphi} = \partial_{p_\varphi} H$$

$$\dot{p}_\psi = -\partial_\psi H, \quad \dot{\psi} = \partial_{p_\psi} H$$

with Hamiltonian function of the form

$$H = -\frac{p_\varphi^2}{2} - \frac{p_\psi^2}{2} + V(\varphi(x), \psi(x)).$$

Here

$$V(\varphi, \psi) = \lambda a_1 \sum_{k=1}^{N} \frac{q_k}{l_k} \int_0^{al_k\varphi} \int_{R^2} A(s, \psi)ds + \lambda a_2 \sum_{k=1}^{N} \frac{q_k}{l_k} \int_0^{bk_k\psi} \int_{R^2} A(\varphi, s)ds.$$

The proof follows from lemma 2.2 (p.1152) of work [4].

Lemma 7. *Assume* **II**. *Then matrix* L_1 *in (38) has one positive eigenvalue*

$$\chi_+ = \mu T_1 + 0(1)$$

and one negative

$$\chi_- = \eta \frac{T_1 T_4 - T_2 T_3}{T_1} \epsilon + O(\epsilon), \quad \eta = \frac{4\pi \mid q \mid}{m} > 0 \tag{44}$$

at $\epsilon \triangleq \frac{1}{c^2} \to 0$.

Eigenvalue χ_- induces the eigenvactors of matrices L_1 and L_1' respectively

$$\begin{bmatrix} c_1 \\ \\ c_2 \end{bmatrix} = \begin{bmatrix} -\frac{T_2}{T_1} \\ \\ 0 \end{bmatrix} + O(\epsilon), \quad \begin{bmatrix} c_1^* \\ \\ c_2^* \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 1 \end{bmatrix} + O(\epsilon).$$

The proof see in [14].

Consider the calculation of bifurcation points λ_0 of equation (36). Setting in (36) $\lambda = \lambda_0 + \epsilon$, we consider the equation

$$(L_0 - (\lambda_0 + \epsilon)L_1 u - (\lambda_0 + \epsilon)r(u) = 0 \tag{45}$$

in neighbourhood of point λ_0. Let $T_2 \neq 0$ and $T_3 \neq 0$, or $T_2 = T_3 = 0$. With the purpose of symmetrization of system at $T_2 \neq 0$ and $T_3 \neq 0$ having multiplicated both parts of (45) on matrix

$$M = \begin{pmatrix} 1 & 0 \\ \\ 0 & \tilde{a} \end{pmatrix}, \quad \text{where } \tilde{a} \overset{\triangle}{\equiv} \frac{\mu T_2}{\nu T_3} \neq 0,$$

we write (45) as

$$Bu = \epsilon B_1 u + (\lambda_0 + \epsilon)\Re(u). \tag{46}$$

Here $B = M(L_0 - \lambda_0 L_1)$; $\Re(u) \overset{\triangle}{=} Mr(u) \overset{\triangle}{=} (r_1(u), r_2(u))$; $B_1 \in L(E \to E)$ be Fredholm self- adjoint operator. If $A_s = A_s(al_s\varphi + bk_k\psi)$, then

$$\frac{\partial A_s}{\partial y} = A_s' b, \quad \frac{\partial A_s}{\partial x} = A_s' a, \quad \tilde{a} = \mu b/(\nu \frac{d^2}{2\alpha a}), \quad \beta_s = \frac{b}{a}\frac{d_s}{2\alpha_s}.$$

In expansion (39)

$$\varrho_{is} = \frac{q_s}{i!} A_s^{(i)}(al_s\varphi^0 + bk_s\psi^0)(al_s u_1 + bk_s u_2)^i.$$

Thus, in this case $\frac{\partial r_1}{\partial u_2} = \frac{\partial r_2}{u_1}$ matrix $\Re_u(u)$ will be symmetric for $\forall u$ and operator $\Re_u : E \to E$ is self-adjoint for $\forall u$.

Remark 5. If $T_2 = T_3 = 0$, then we put $\tilde{a} = 1$. If $T_2 = 0$, $T_3 \neq 0$ or $T_3 = 0$, $T_2 \neq 0$, then the problem (36) has not the property of symmetrization and we should work with (45). In this case for study of the problem on bifurcation point we may use our results from [13].

Let μ is eigenvalue of Dirichlet problem

$$-\triangle e = \mu e \;\; e \mid_{\partial D} = 0 \tag{47}$$

and $\{e_1, \ldots, e_n\}$ be orthonormalized basis in a subspace of eigenfunctions. Denote by $c_- = (c_1, c_2)'$ the eigenvector of matrix L_1, which corresponds to eigenvalue $\chi_- < 0$.

Lemma 8. *Let $\lambda_0 = -\mu/\chi_-$. Then $\lambda_0 > 0$, $dim N(B) = n$ and the system $\{\mathbf{e}_i\}_{i=1}^n$, where $\mathbf{e}_i = c_- e_i$ forms basis in a subspace $N(B)$.*

Proof. Let us introduce matrix of columns Λ, which are the eigenvectors of matrix L_1 corresponding to eigenvalues χ_-, χ_+. Moreover,

$$\Lambda^{-1} L_1 \Lambda = \begin{pmatrix} \chi_- & 0 \\ & \\ 0 & \chi_+ \end{pmatrix}, \;\; L_0 \Lambda = \Lambda L_0$$

and equation $Bu = 0$ by change $u = \Lambda U$ will be transformed to the form

$$M[L_0 \Lambda U - \lambda_0 L_1 \Lambda U] = M[\Lambda(L_0 U - \lambda_0 \Lambda^{-1} L_1 \Lambda U)] = 0.$$

Hence, from here follows that the linear system (41) is decomposed onto two linear elliptical equations

$$\triangle U_1 - \lambda_0 \chi_- U_1 = 0, \;\; U_1 \mid_{\partial D} = 0, \quad \triangle U_2 - \lambda_0 \chi_+ U_2 = 0, \;\; U_2 \mid_{\partial D} = 0, \tag{48}$$

where $\lambda_0 \chi_- = -\mu$, $\lambda_0 \chi_+ > 0$. From (47) follows that $\mu \in \sigma(-\triangle)$. Hence, $U_1 = \sum_{i=1}^n \alpha_i e_i$, $\alpha_i - const$, $U_2 = 0$ and

$$\begin{vmatrix} u_1 \\ \\ u_2 \end{vmatrix} = \Lambda U = \begin{vmatrix} c_{1-} & c_{1+} \\ & \\ c_{2-} & c_{2+} \end{vmatrix} = \begin{vmatrix} U_1 \\ \\ 0 \end{vmatrix} = \begin{vmatrix} c_{1-} \\ \\ c_{2-} \end{vmatrix} \sum_{i=1}^n \alpha_i e_i.$$

Let us construct Lyapunov-Schmidt BEq for equation (46).

Without loss of generality we assume that the eigenvector c_{1-} of matrix L_1 is chosen such that $\chi_-(c_{1-}^2 + F c_{2-}^2) = 1$, where $F = \frac{\mu T_2}{\nu T_3}$. Then the system of vectors $\{B_1 \mathbf{e}_i\}_{i=1}^n$ is biorthogonal to $\{\mathbf{e}_i\}_{i=1}^n$. Thus, operator

$$\breve{B} = B + \sum_1^n < \cdot, \gamma_i > \gamma_i$$

with $\gamma_i \stackrel{\triangle}{=} B_1 \mathbf{e}_i$ has inverse bounded $\Gamma \in L(E \to E)$, $\Gamma = \Gamma^*$, $\Gamma \gamma_i = \mathbf{e}_i$.

Rewrite (46) as the system

$$(\breve{B}-\epsilon B_1)u=(\lambda_0+\epsilon)\Re(u)+\sum_i \xi_i\gamma_i \tag{49}$$

$$\xi_i=<u,\gamma_i>,\quad i=1,\dots,n. \tag{50}$$

By the theorem on inverse operator we have from (49)

$$u=(\lambda_0+\epsilon)(I-\epsilon\Gamma B_1)^{-1}\Gamma\Re(u)+\frac{1}{1-\epsilon}\sum_{i=1}^{n}\xi_i\mathbf{e}_i. \tag{51}$$

From (50) we have

$$\frac{\epsilon}{1-\epsilon}\xi_i+\frac{\lambda_0+\epsilon}{1-\epsilon}<\Re(u),\mathbf{e}_i>=0, \tag{52}$$

where $\Re(u)=\Re_l(u)+\Re_{l+1}(u)+\dots$. Because of the theorem on implicit operator, equation (51) has unique solution for sufficiently small ϵ, $\mid\xi\mid$.

$$u=u_1(\xi\mathbf{e},\epsilon)+(\lambda_0+\epsilon)(I-\epsilon\Gamma B_1)^{-1}\Gamma\{u_l(\xi\mathbf{e},\epsilon)+u_{l+1}(\xi\mathbf{e},\epsilon)+\dots\}. \tag{53}$$

Here

$$u_1(\xi\mathbf{e},\epsilon)=\frac{1}{1-\epsilon}\sum_{i=1}^{n}\xi_i\mathbf{e}_i,$$

$$u_l(\xi\mathbf{e},\epsilon)=\Re_l(u_1(\xi\mathbf{e},\epsilon)),$$

$$u_{l+1}(\xi\mathbf{e},\epsilon)=\Re_{l+1}(u_1(\xi\mathbf{e},\epsilon))+$$

$$+\begin{cases}0, & l\geq 2\\ \Gamma\Re_2'(u_1(\xi\mathbf{e},\epsilon))(\lambda_0+\epsilon)(I-\epsilon\Gamma B_1)^{-1}\Gamma u_2(\xi\mathbf{e},\epsilon), & l=2\end{cases}$$

and etc. Substituting the solution (53) into (52) we obtain desired BEq

$$\mathbf{L}(\xi,\epsilon)=0 \tag{BEq}$$

with $\mathbf{L}=(L^1,\dots,L^n)$,

$$\mathbf{L}^i=\frac{\epsilon}{1-\epsilon}\xi_i+\frac{\lambda_0+\epsilon}{(1-\epsilon)^{l+1}}[<\Re_l(\xi\mathbf{e},\mathbf{e}_i)>+\frac{1}{1-\epsilon}<\Re_{l+1}(\xi\mathbf{e},\mathbf{e}_i)>]+$$

$$\begin{cases}0, \quad l>2\\ \frac{\lambda_0+c}{(1-\varepsilon)^4}<\Re_2'(\xi\mathbf{e}(I-\varepsilon\Gamma B_1)^{-1}\Gamma\Re_2(\xi\mathbf{e}),\mathbf{e}_i>, \quad l=2\end{cases}+r_i(\xi,\varepsilon),$$

$r_i = o(|\ \xi\ |^{l+1})$, $i = 1, \ldots, n$. If $\mathbf{L}(\xi, \varepsilon) = gradU(\xi, \varepsilon)$, then we call BEq potential. In potential case matrix $\mathbf{L}_\xi(\xi, \varepsilon)$ is symmetric.

Let in (46) $f_i = f_i(al_i\varphi + bk_i\psi)$, $i = 1, \ldots, N$. Then from explained above matrix $\Re_u(u)$ will be symmetric at $\forall u$ and we have the following statement:

Lemma 9. *Let conditions* **C)**, **D)**, **I-II** *and* $\lambda_0 = -\mu/\chi_-$ *hold. Then equation (46) possesses so much small solutions* $u \to 0$ *at* $\lambda \to \lambda_0$, *as small solutions* $\xi \to 0$ *possesses BEq at* $\varepsilon \to 0$. *If in system (32)* $A_i = A_i(al_i\varphi + bk_i\psi)$, $i = 1, \ldots, N$; a, b*--const, then BEq will be potential.*

Principal theorem. *Let* $N \geq 3$. *Let conditions* **C**, **D**, **I-II** *and* $\lambda_0 = -\mu/\chi$ *are valid, where* μ *is* n *multiple eigenvalue of Dirichlet problem (47). Number* χ_- *see in (44). If* n *is odd, or distribution function has the form* $f_i = f_i(a(-\alpha_i v^2 + \varphi_i) + b((d_i, v) + \psi_i))$, $i = 1, \ldots, N$, *then* λ_0 *be a bifurcation point of VM system (1)-(2) with conditions* (3)-(4).

Proof. Case 1. Let n is odd. Then in BEq

$$\triangle(\varepsilon) \equiv \det |\frac{\partial L_k}{\xi_i}(0, \varepsilon)|_{i,k=1}^n = (\frac{\varepsilon}{1-\varepsilon})^n.$$

Since n is odd, then $\triangle(\varepsilon) > 0$ for $\varepsilon \in (0,1)$, and $\triangle(\varepsilon) < 0$ for $\varepsilon \in (-1, 0)$ and the statement of theorem follows from theorem 1.

Case 2. Let $f_i = f_i(a(-\alpha_i v^2 + \varphi_i) + b((d_i, v) + \psi_i))$. Then BEq is potential, moreover

$$\frac{L_k(0, \varepsilon)}{\partial \xi_i} = \frac{\varepsilon}{1-\varepsilon}\delta_{ik}, \quad i, k = 1, \ldots, n.$$

Hence, all eigenvalues of matrix $\parallel \frac{L_k(0,\varepsilon)}{\partial \xi_i} \parallel$ are positive at $\varepsilon > 0$ and are negative at $\varepsilon < 0$. Thus, the validity of the theorem in case 2 follows from theorem2.

Possible generalizations. The distributions functions f_i in VM system depend not only upon λ, but also on parameters α_i, d_i, k_i, l_i. It seems interest to investigate a behaviour of solutions of (1)-(2) with conditions (3), (4) depending from these parameters. Applying theorems 1, 2 and their corollaries in the present paper, we can prove the existence theorems of points and surfaces of bifurcation for this more complicated case. We shall consider this problem in the following paper.

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