POSITIVE SOLUTIONS OF NONLINEAR SINGULAR BOUNDARY-VALUE PROBLEM OF MAGNETIC INSULATION

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Abstract: On the basis of generalization of upper and lower solution method to the singular two-point boundary value problems, the existence theorem of solutions for the system, which models a process of magnetic insulation in plasma is proved.

keywords:singular boundary value problem, upper and lower solution, magnetic insulation.

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1. Introduction.

Electron[1] transport in high energy devices such as vacuum diodes exhibits many non-linear phenomena due to the extremely high applied voltages. One of these effects is the saturation of the current due to the self-consistent electric and magnetic field. Langmuir and Compton [1] have investigated this phenomenon the first and established explicit formulae for the saturation current in the plane diode case, and approximate ones in the cylindrical and spherical diode cases. They assumed that the current saturates at a maximal value determined by the condition that the electric field vanishes at the emission cathode. This condition is referred to as the Child-Langmuir condition and the diode is said to operate under a space charge limited or a Child-Langmuir regime.

Investigation of mathematic models of magnetic insulation has been started by P.Degond, N.Ben Abdallah and F.Mehats in 1995 year. In 1996 P.Degond has put to the author of this Appendix the problem on existence of solutions of limit system (I) and its generalization to the problem with free boundary.

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The effect of magnetic insulation consists in that the electrons emitted from cathode cannot reach the anode due to the extremely high applied electric and magnetic field; they are reflected by the magnetic forces back to the cathode. Thus there is electronic layer outside of which electromagnetic field is equal to zero (see Longmuir and Compton [1]). Here two basic regimes are possible: the first, when electrons reach the anode − "noninsulated" diode and the second one, when electrons rotate back to the cathode − "insulated" diode. The regime of "noninsulated" diode is described by the following nonlinear two-point boundary value problem

$$\frac{d^2\varphi}{dx^2} = j_x \frac{1+\varphi(x)}{\sqrt{(1+\varphi(x))^2 - 1 - a(x)^2}} \triangleq F(\varphi, a); \quad \varphi(0) = 0, \quad \varphi(1) = \varphi_L,$$

$$\tag{I}$$

$$\frac{d^2 a}{dx^2} = j_x \frac{a(x)}{\sqrt{(1+\varphi(x))^2 - 1 - a(x)^2}} \triangleq G(\varphi, a); \quad a(0) = 0, \quad a(1) = a_L,$$

where $j_x > 0$, $x \in [0,1]$; φ is the potential of electric field and the potential of magnetic field is a.

Our main goal consists in search of positive solutions of system (I) that is $\varphi > 0$, $a > 0$ and their dependences upon parameter j_x. Here there are some interesting questions about solvability of this problem, because the system (I) is singular in zero for $\varphi = 0$ andf in this connection, we can not say about properties of monotonicity of right parts on the interval $\varphi \in [0, \infty)$ and, hence, about Lipschitz condition. The problem (I) has no a property of quasimonotonicity in cone. Thus a standard upper and lower solution method, developed for the systems of semilinear elliptic equations in partially ordered Banach space (see Amann [3]), does not work. In spite of this fact, we show the existence of lower and upper solutions of problem (I) without conditions of local Lipschitz continuity and quasimonotonicity using sufficiently simple technics. To this purpose, we modify the McKenna and Walter [1] theorem of existence of lower and upper solutions for arbitrary elliptic systems

$$\triangle u + f(x,u) = 0 \quad \Omega, \quad u = 0 \quad \partial\Omega, \tag{1.1}$$

where $u = (u_1, \ldots, u_n)$, $f = (f_1, \ldots, f_n)$ are $n -$ vectors; Ω is an open bounded subset of R^M with smooth boundary $\partial\Omega$, and $f(x,u)$ is uniformly Hö lder continuous (with exponent α) in x and Lipschitz continuous in u.

The outline of the Appendix is as follows. In Subsection 2, we give the statement and the derivation of system (I). The model that we shall consider is the 1.5 dimensional stationary *relativistic Vlasov-Maxwell(VM) system.* Introducing a small parameter $\epsilon > 0$ and the spacial dimensionless variables, into the VM system, we obtain the singular perturbation problem. Next, using some invariants of the electron motion in the limit $\epsilon \to 0$, give system (I).

In Section 3 we will prove Theorem 3.1 and Propositions 3.1, 3.2 on the existence of semitrivial solutions of problem (I) by upper and lower solution method. The estimations to the value of electrostatic potential on the anode φ_L and the current j_x are obtained. In Section 4 we formulate the principal Theorem 4.1 on the existence of positive solutions of problem (I) and the estimation to the value of magnetic field on the anode a_L is given.

We note that system (I) was studied in Abdallah, Degond and Mehats [2] by a shooting method with $\beta = a'(0)$ and j_x as shooting parameters. The strategy is: given the values of β and j_x, solve (I) with the Cauchy conditions $\varphi(0) = 0$, $a(0) = 0$, $\varphi'(0) = 0$, $a'(0) = \beta$, and then adjust the values in order to fulfill the conditions $\varphi(1) = \varphi_L$ and $a(1) = a_L$.

2. Setting of the problem and derivation of system (I).

We consider a plane diode consisting of two perfectly conducting electrodes, a cathode $(X = 0)$ and anode $(X = L)$ supposed to be infinite planes, parallel to (Y, Z) (Fig. 1).

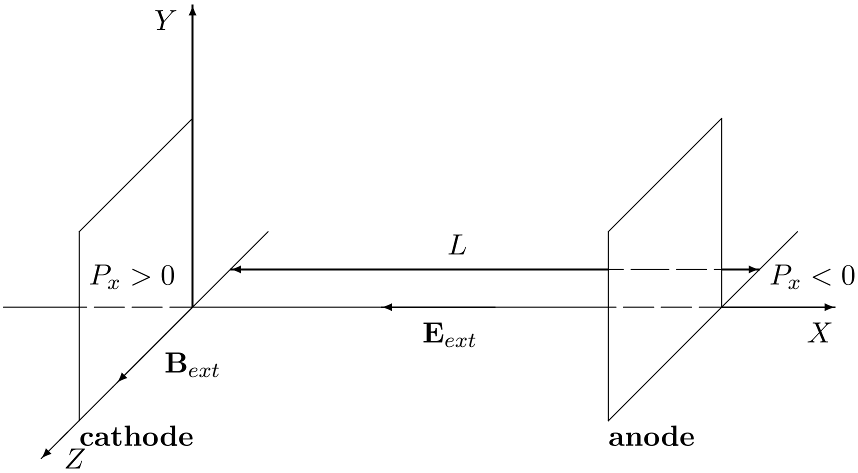

Figure 1: circuit of diode

The electrons, with charge $-e$ and mass m, are emitted at the cathode and submitted to an applied electromagnetic field

$$\mathbf{E}_{ext} = E_{ext}\mathbf{X}, \quad \mathbf{B}_{ext} = B_{ext}\mathbf{Z}$$

such that $E_{ext} \leq 0$ and $B_{ext} \geq 0$.

We shall assume that the electron distribution function F does not depend on Y and that the flow is stationary and collisionless. The system is then described by the so called 1.5 dimensional VM model 1.5

$$V_X \frac{\partial F}{\partial X} + e\left(\frac{d\Phi}{dX} - V_Y \frac{dA}{dX}\right)\frac{\partial F}{\partial P_X} + eV_X \frac{dA}{dX}\frac{\partial F}{\partial P_Y} = 0, \tag{2.1}$$

$$\frac{d^2\Phi}{dX^2} = \frac{e}{\epsilon_0} N(X), \qquad X \in (0, L), \tag{2.2}$$

$$\frac{d^2 A}{dX^2} = -\mu_0 J_Y(X), \quad X \in (0, L) \tag{2.3}$$

subject to the following boundary conditions:

$$F(0, P_X, P_Y) = G(P_X, P_Y), \quad P_X > 0, \tag{2.4}$$

$$F(L, P_X, P_Y) = 0, \qquad P_X < 0, \tag{2.5}$$

$$\Phi(0) = 0, \quad \Phi(L) = \Phi_L = -LE_{ext}, \tag{2.6}$$

$$A(0) = 0, \quad A(L) = A_L = LB_{ext}, \tag{2.7}$$

where formulas (2.4) and (2.5) describe the injection profile at the cathode and at the anode, respectively, $E = -d\Phi/dX$, $B = -dA/dX$. The relationship between momentum an velocity is then given by the relativistic relations

$$\mathbf{V}(\mathbf{P}) = \frac{\mathbf{P}}{\gamma m}, \quad \gamma = \sqrt{1 + \frac{|\mathbf{P}|^2}{m^2c^2}},$$

$$\mathbf{V} = (V_X, V_Y), \quad \mathbf{P} = (P_X, P_Y), \quad |\mathbf{P}|^2 = P_X^2 + P_Y^2,$$

or

$$\mathbf{V}(\mathbf{P}) = \nabla_{\mathbf{P}} E(\mathbf{P}),$$

where E is the relativistic kinetic energy and c is the speed of light.

In system (2.1)-(2.3), the macroscopic quantities, namely the particle density N; X and Y components of the current density J_X, J_Y, are respectively given by the following formulas

$$N(X) = \int_{R^2} F(X, P_X, P_Y) dP_X dP_Y, \tag{2.8}$$

$$J_X = -e \int_{R^2} V_X(\mathbf{P}) F(X, P_X, P_Y) dP_X dP_Y, \tag{2.9}$$

$$J_Y(X) = -e \int_{R^2} V_Y(\mathbf{P}) F(X, P_X, P_Y) dP_X dP_Y. \tag{2.10}$$

Here, ϵ_0 and μ_0 are respectively the vacuum permittivity and permeability.

The 1.5 model describes two principal regimes. For a strong applied magnetic field , electrons do not reach the anode and come back to the cathode leading to a vanishing J_X component of current density; our model is fully rigorous in this case. When the applied magnetic field is not strong enough to insulate the diode, J_X does not vanish and our model can be viewed as an approximate of the Maxwell equations.

Similarly to (2.8)-(2.10), we define the moments associated with the incoming particle distribution function by

$$N^G = \int_{R^2_+} G(P_X, P_Y) dP_X dP_Y, \tag{2.11}$$

$$J_X^G = -e \int_{R^2_+} V_X(\mathbf{P}) G(P_X, P_Y) dP_X dP_Y, \tag{2.12}$$

$$J_Y^G = -e \int_{R^2_+} V_Y(\mathbf{P}) G(P_X, P_Y) dP_X dP_Y, \tag{2.13}$$

$$T^G = \int_{R^2_+} E(\mathbf{P}) G(P_X, P_Y) dP_X dP_Y, \tag{2.14}$$

where $R^2_+ = \{(P_X, P_Y) \in R^2, P_X > 0\}$, see Fig. 1 and the thermal emission velocity is $V^G = \sqrt{\frac{T^G}{mN^G}}$. The quantities (2.11)-(2.14), respectively define the incoming particle density, X and Y components of the incoming current density and incoming particle kinetic energy.

In order to get a better insight in the behaviour of the diode, we write the model (2.1)-(2.7) in dimensionless variables. Following of Degond and Raviart [6, 7], we introduce the following units respectively for position, velocity, momentum, electrostatic potential, vector potential, particle density, current and distribution function:

$$\bar{X} = L,\ \ \bar{V} = c,\ \ \bar{P} = mc,\ \ \bar{E} = mc^2,\ \ \bar{\Phi} = \frac{mc^2}{e},$$

$$\bar{A} = \frac{mc}{e},\ \ \bar{N} = \frac{\epsilon_0 \bar{\Phi}}{x\bar{X}^2},\ \ \bar{J} = -ec\bar{N},\ \ \bar{F} = \frac{\bar{N}}{\bar{P}^2}$$

and the corresponding dimensionless variables

$$x = X/\bar{X},\ \ \mathbf{p} = \frac{\mathbf{P}}{\bar{P}} = (p_x, p_y), \mathbf{v} = (v_x, v_y) = \frac{\mathbf{V}}{\bar{V}} = \mathbf{p}/\sqrt{1+\mathbf{p}^2},$$

$$\Xi = E/\bar{E} = \sqrt{1+\bar{p}^2} - 1,\ \ \varphi = \Phi/\bar{\Phi},\ \ a = A/\bar{A},\ \ n = N/\bar{N},$$

$$j = J/\bar{J},\ \ f = F/\bar{F}.$$

Let the diode is controlled in the Child-Langmuir regime. In such a situation, the thermal velocity V_G is much smaller than the typical drift velocity supposed to be of the order of the speed of light c. Letting $\varepsilon = \frac{V_G}{c}$, we shall assume that

$$f(0, p_x, p_y) = g^\varepsilon(p_x, p_y) = \frac{1}{\varepsilon^3} g(\frac{p_x}{\varepsilon}, \frac{p_y}{\varepsilon}),\ \ p_x > 0,$$

where g is a given profile. The dimensionless system reads

$$v_x \frac{\partial f^\varepsilon}{\partial x} + \left(\frac{d\varphi^\varepsilon}{dx} - v_y \frac{da^\varepsilon}{dx}\right) \frac{\partial f^\varepsilon}{\partial p_x} + v_x \frac{da^\varepsilon}{dx} \frac{\partial f^\varepsilon}{\partial p_y} = 0, \tag{2.15}$$

$$(x, p_x, p_y) \in (0,1) \times R^2,$$

$$\frac{d^2\varphi^\varepsilon}{dx^2} = n^\varepsilon(x),\ \ x \in (0,1), \tag{2.16}$$

$$\frac{d^2 a^\varepsilon}{dx^2} = j_y^\varepsilon(x),\ \ x \in (0,1), \tag{2.17}$$

$$n^\varepsilon(x) = \int_{R_+^2} f^\varepsilon(x, p_x, p_y) dp_x dp_y, \tag{2.18}$$

$$j_y^\varepsilon(x) = \int_{R_+^2} v_y f^\varepsilon(x, p_x, p_y) dp_x dp_y =$$

$$\int_{R_+^2} \frac{p_y}{\sqrt{1+|p|^2}} f^\varepsilon(x, p_x, p_y) dp_x dp_y, \tag{2.19}$$

$$f^\varepsilon(0, p_x, p_y) = g^\varepsilon(p_x, p_y) = \frac{1}{\varepsilon^3} g(\frac{p_x}{\varepsilon}, \frac{p_y}{\varepsilon}),\ \ p_x > 0, \tag{2.20}$$

$$f^\varepsilon(1, p_x, p_y) = 0,\ \ p_x < 0, \tag{2.21}$$

$$\varphi^\varepsilon(0) = 0,\ \ \ \varphi^\varepsilon(1) = \varphi_L, \tag{2.22}$$

$$a^\varepsilon(0) = 0, \quad a^\varepsilon(1) = a_L. \tag{2.23}$$

To derive the limit model (I) at $\varepsilon \to 0$, we consider the various invarians of the problem. The following two quantities are constants of motion

$$W^\varepsilon(x,p) = \Xi(p) - \varphi^\varepsilon(x) - \text{the electron energy}, \tag{2.24}$$

$$\mathcal{P}_y^\varepsilon(x,p) = p_y - a^\varepsilon(x) - \text{the canonical momentum}, \tag{2.25}$$

which means that on each electron trajectory (in the phase space), the above quantities are constant. Let us denote f, n, a, j, φ ... the limit as ϵ tends to zero f^ε, n^ε, ... Since, in the limit $\varepsilon = 0$, electrons are injected with zero velocity, it is readily seen that the electron energy W and canonical momentum $\mathcal{P}_y$ simultaneously vanish. Consequently,

$$p_y(x) = a(x),$$
$$(p_x(x))^2 = (1+\varphi(x))^2 - 1 - (a(x))^2$$

and the following identities hold:

$$v_x(x) = \frac{p_x(x)}{\sqrt{1+\mathbf{p}^2(x)}} = \frac{p_x(x)}{1+\varphi(x)},$$

$$v_y(x) = \frac{v_y(x)}{\sqrt{1+\mathbf{p}^2(x)}} = \frac{a(x)}{1+\varphi(x)}.$$

Let us now define the *effective potential* by

$$\Theta(x) = (1+\varphi(x))^2 - 1 - (a(x))^2. \tag{2.26}$$

Electrons do not enter the diode unless the effective potential Θ is nonnegative in the vicinity of the cathode. Therefore, we always have $\Theta'(0) \geq 0$. Let Θ_L be the value of Θ at the anode

$$\Theta_L = (1+\varphi_L)^2 - 1 - a_L^2. \tag{2.27}$$

If $\Theta_L < 0$, electrons cannot reach the anode $x = 1$; they are reflected by the magnetic forces back to the cathode and the diode is said to be *magnetically insulated.* If Θ is nonnegative, then all electrons are reached the anode and the diode is said to be *noninsulated.*

The aim of this chapter is to give an analysis of noninsulated regime. We assume that

$$\forall x \in (0,1], \quad \Theta(x) > 0, \quad \Theta(1) - \Theta(0) = \Theta_L > 0.$$

The last one denotes, a phase portrait (x, p_x) of electron trajectory has the form

Since no electron is injected at the anode, j_x^- vanishes. Hence

$$j_x = j_x^+ = \int_{R_+^2} v_x f(x, p_x, p_y) dp_x dp_y$$

and the distribution function is that of a monokinetic beam issued from the cathode $x = 0$ with vanishing initial velocity

$$f(x, \mathbf{P}) = n(x)\delta\Big(p_x - \sqrt{\Theta(x)}\Big)\delta(p_y - a(x)).$$

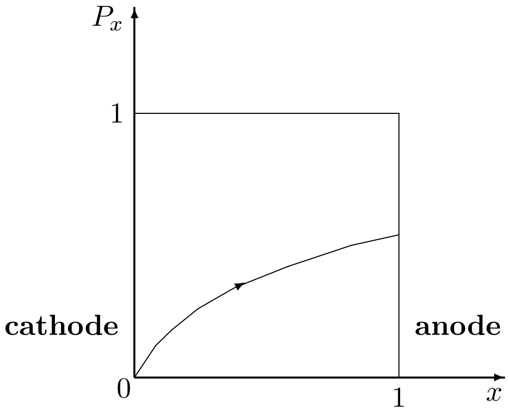

Figure 2:

Therefore

$$n(x) = \frac{j_x}{v_x(x)} = j_x \frac{1+\varphi(x)}{\sqrt{\Theta(x)}}, \quad j_y(x) = n(x)v_y(x) = j_x \frac{a(x)}{\sqrt{\Theta(x)}}.$$

Inserting these expressions into Poisson's and Ampere's equations (2.2), (2.3) gives

$$\frac{d^2\varphi}{dx^2}(x) = j_x \frac{1+\varphi(x)}{\sqrt{(1+\varphi(x))^2 - 1 - (a(x))^2}}, \quad \varphi(0) = 0, \quad \varphi(1) = \varphi_L,$$

$$(I)$$

$$\frac{d^2 a}{dx^2}(x) = j_x \frac{a(x)}{\sqrt{(1+\varphi(x))^2 - 1 - (a(x))^2}}, \quad a(0) = 0, \quad a(1) = a_L.$$

In system (I) the unknowns are the electrostatic potential φ, the magnetic potential a and the current j_x (which does not depend on x).

3. Existence of semitrivial solutions of problem (I).

Let us introduce the definition of cone in a Banach space X.

Definition 3.1: Let X be a Banach space. A nonempty convex closed set $P \subset X$ is called a *cone*, if it satisfies the conditions:
(i) $x \in P$, $\lambda \geq 0$ implies $\lambda x \in P$;
(ii) $x \in P$, $-x \in P$ implies $x = \mathcal{O}$, where $\mathcal{O}$ denotes zero element of X. Here $\leq$ is the order in X induced by P, i.e., $x \leq y$ if and only if $y - x$ is an element of P.

We will denote $[x, y]$ the closed order interval between x and y, i.e.,

$$[x, y] = \{z \in X : x \leq z \leq y\}. \tag{3.1}$$

We will also assume that the cone P is normal in X, i.e., order intervals are norm bounded.

In X

$$X \equiv \{(u, v) : \ u, v \in C^1(\bar{\Omega}), u = v = 0\}$$

we introduce the norm $|U|_X = |u|_{C^1} + |v|_{C^1}$, and the norm $|U|_X = |u|_\infty + |v|_\infty$ in C, where $U = (u, v)$. Here a cone P is given by

$$P = \{(u, v) \in X : \ u \geq 0, v \geq 0 \ \text{ for all } \ x \in \Omega\}. \tag{3.2}$$

So, if $u \neq 0$, $v \neq 0$ belong to P, then $-u, -v$ does not belong. We will work with classical spaces on the intervals $\bar{I} = [a, b]$, $\hat{I} =]a, b]$, $I = (a, b)$:
$C(\bar{I})$ with norm $\| u \|_\infty = max\{|u(x)| : \ x \in \bar{I}\}$;
$C^1(\bar{I}) = \| u \|_\infty + \| u' \|_\infty$;
$C_{loc}(I)$, which contains all functions that are locally absolutely continuous in I. We introduce a space $C_{loc}(I)$ because the problem (I) is singular for $\varphi = 0$. The order $\leq$ in cone P is understood in the weak sense, i.e., y is increasing if $a \leq b$ implies $y(a) \leq y(b)$ and y is decreasing if $a \leq b$ implies $y(a) \geq y(b)$.

Theorem 3.1: (Walter [4]) (comparison principle in cone) *Let $y \in C(\bar{I}) \bigcap C_{loc}(I)$. The function f is defined on $I \times R$. Let $f(x, y)$ is increasing in y function, then*

$$v'' - f(x, v) \geq w'' - f(x, w) \ \ .. \quad I, \tag{3.3}$$

$$v(a) \leq w(a), \quad v(b) \leq w(b)$$

implies

$$v \leq w \quad on \ \ \bar{I}.$$

Remark 3.1: Let $f(x, y)$ is decreasing, then Theorem 3.1 remains without changes, if both parts of (3.3) multiply onto -1.

For the convenience of defining an ordering relation in cone P, we make a transformation for the problem (I). Let $F(\varphi, a)$ and $G(\varphi, a)$ be defined by (I). Then throuth the transformation $\varphi = -u$ the problem (I) is reduced to the form

$$-\frac{d^2u}{dx^2} = j_x \frac{1-u}{\sqrt{(1-u)^2 - 1 - a^2}} \triangleq \tilde{F}(j_x, u, a), \ \ u(0) = 0, \ \ u(1) = \varphi_L,$$

$$\frac{d^2a}{dx^2} = j_x \frac{a}{\sqrt{(1-u)^2 - 1 - a^2}} \triangleq \tilde{G}(j_x, u, a), \ \ a(0) = 0, \ \ a(1) = a_L. \tag{II}$$

We note that all solutions of the problem (I), as well the problem (II), are symmetric with respect to the transformation of sign for the magnetic potential a : $(\varphi, a) = (\varphi, -a)$ or the same $(u, a) = (u, -a)$. Thus we must search only positive solutions $\varphi > 0$, $a > 0$ in cone P or only negative ones: $\varphi < 0$, $a < 0$. Thanks to the symmetry of problem it is equivalently and does not yields the extension of the types of sign-defined solutions of the problem (I) (respect. (II)). Once more, we note that introduction of negative electrostatic potential in problem (II) is connected with more convenient relation between order in cone and positiveness of Green function for operator $-u''$ that we use below.

Definition 3.2: A pair $[(\varphi_0, a_0), (\varphi^0, a^0)]$ is called
a) sub-super solution of the problem (I) relative to P, if the following conditions are satisfied

$$\begin{cases} (\varphi_0, a_0) \in C_{loc}(I) \bigcap C(\bar{I}) \times C_{loc}(I) \bigcap C(\bar{I}), \\ (\varphi^0, a^0) \in C_{loc}(I) \bigcap C(\bar{I}) \times C_{loc}(I) \bigcap C(\bar{I}) \end{cases}; \tag{3.4}$$

$$\varphi_0'' - j_x \frac{1+\varphi_0}{\sqrt{(1+\varphi_0)^2 - 1 - a^2}} \triangleq F(\varphi_0, a) \le 0 \ \text{ in } \ I,$$

$$(\varphi^0)'' - j_x \frac{1+\varphi^0}{\sqrt{(1+\varphi^0)^2 - 1 - a^2}} \triangleq F(\varphi^0, a) \ge 0 \ \text{ in } \ I \ \ \forall a \in [a_0, a^0]; \tag{3.5}$$

$$a_0'' - j_x \frac{a_0}{\sqrt{(1+\varphi)^2 - 1 - a_0^2}} \triangleq G(\varphi, a_0) \le 0 \ \text{ in } \ I,$$

$$(a^0)'' - j_x \frac{a^0}{\sqrt{(1+\varphi)^2 - 1 - (a^0)^2}} \triangleq G(\varphi, a^0) \ge 0 \ \text{ in } \ I \ \ \forall \varphi \in [\varphi_0, \varphi^0]; \tag{3.6}$$

$$\varphi_0 \le \varphi^0, \quad a_0 \le a^0 \ \text{ in } \ I \tag{3.7}$$

and on the boundary

$$\varphi_0(0) \le 0 \le \varphi^0(0), \quad \varphi_0(1) \le \varphi_L \le \varphi^0(1),$$

$$a_0(0) \le 0 \le a^0(0), \quad a_0(1) \le a_L \le a^0(1); \tag{3.8}$$

b) sub-sub solution of the problem (I) relative to P, if a condition (3.4) is satisfied and

$$\varphi_0'' - F(j_x, \varphi_0, a_0) \le 0 \ \text{ in } \ I,$$

$$a_0'' - G(j_x, \varphi_0, a_0) \le 0 \ \text{ in } \ I \tag{3.9}$$

and on the boundary

$$\varphi_0(0) \le 0, \ \ \varphi_0(1) \le \varphi_L, \quad a_0(0) \le 0, \ \ a_0(1) \le a_L. \tag{3.10}$$

Remark 3.2: In Definition 3.2 the expressions with square roots we take by modulus $|(1+\varphi)^2 - 1 - a^2|$.

By analogy with (3.9), (3.10), we may introduce the definition of super-super solution in cone.

Definition 3.3: The functions $\Phi(x, x_{a_i}, j_x)$, $\Phi_1(x, x_{\varphi_j}, j_x)$ we shall call a *semitrivial solutions* of the problem (I), if $\Phi(x, x_{a_i}, j_x)$ is a solution of the scalar boundary value problem

$$\varphi'' = F(j_x, \varphi, x_{a_i}) = j_x \frac{1+\varphi}{\sqrt{(1+\varphi)^2 - 1 - (x_{a_i})^2}}, \tag{III}$$

$$\varphi(0) = 0, \ \ \varphi(1) = \varphi_L,$$

and $\Phi_1(x, x_{\varphi_j}, j_x)$ is a solution of the scalar boundary value problem

$$a'' = G(j_x, x_{\varphi_j}, a) = j_x \frac{a}{\sqrt{(1+x_{\varphi_j})^2 - 1 - a^2}}, \tag{IV}$$

$$a(0) = 0, \;\; a(1) = a_L.$$

Here x_{a_i}, $i = 1, 2, 3$ and x_{φ_j}, $j = 1, 2$ are respectively, the *indicators* of semitrivial solutions $\Phi(x, x_{a_i}, j_x)$, $\Phi_1(x, x_{\varphi_j}, j_x)$ defined by the following way:
$x_{a_1} = 0$, if $a(x) = 0$;
$x_{a_2} = a^0$, if $a = a^0$ be upper solution of the problem (IV);
$x_{a_3} = a_0$, if $a = a_0$ be lower solution of the problem (IV);
$x_{\varphi_1} = \varphi^0$, if $\varphi = \varphi^0$ be upper solution of the problem (III);
$x_{\varphi_2} = \varphi_0$, if $\varphi = \varphi_0$ be lower solution of the problem (III).

From Definition 3.3, we obtain the following types of scalar boundary value problems for semitrivial (in sense of Definition 3.3) solutions (I) (resp. (II)):

$$\varphi'' = F(\varphi, 0) = j_x \frac{1+\varphi}{\sqrt{(1+\varphi)^2 - 1}}, \;\; \varphi(0) = 0, \;\; \varphi(1) = \varphi_L. \tag{A$_1$}$$

$$\varphi'' = F(\varphi, a^0) = j_x \frac{1+\varphi}{\sqrt{(1+\varphi)^2 - 1 - (a^0)^2}}, \;\; \varphi(0) = 0, \;\; \varphi(1) = \varphi_L. \tag{A$_2$}$$

$$\varphi'' = F(\varphi, a_0) = j_x \frac{1+\varphi}{\sqrt{(1+\varphi)^2 - 1 - (a_0)^2}}, \;\; \varphi(0) = 0, \;\; \varphi(1) = \varphi_L. \tag{A$_3$}$$

$$a'' = G(\varphi^0, a) = j_x \frac{a}{\sqrt{(1+\varphi^0)^2 - 1 - a^2}}, \;\; a(0) = 0, \;\; a(1) = a_L. \tag{A$_4$}$$

$$a'' = G(\varphi_0, a) = j_x \frac{a}{\sqrt{(1+\varphi_0)^2 - 1 - a^2}}, \;\; a(0) = 0, \;\; a(1) = a_L. \tag{A$_5$}$$

We shall find the solutions of problems $(A_1) - (A_3)$ with condition

$$\varphi_0 < \varphi^0,$$

where $\varphi_0(x_{a_1})$, $\varphi^0(x_{a_2})$ are respectively, lower and upper solutions of problem (A_1). The solution (φ, a) of problem (I) should be belong to the interval

$$\varphi \in \Phi(\varphi, 0) \bigcap \Phi(\varphi, a^0) \bigcap \Phi(\varphi, a_0),$$

$$a \in \Phi_1(\varphi^0, a) \bigcap \Phi_1(\varphi_0, a).$$

Moreover, the ordering of lower and upper solutions of problems $(A_1) - (A_3)$ is satisfied

$$\varphi_0(x_{a_1}) < \varphi_0(x_{a_2}) < \varphi_0(x_{a_3}) < \varphi^0(x_{a_2}) < \varphi^0(x_{a_1}).$$

We shall seek the solution of problems $(A_4) - (A_5)$ with condition

$$a_0 < a^0.$$

In this case the following ordering of lower and upper solutions of problems $(A_4) - (A_5)$

$$a_0(x_{\varphi_1}) < a_0(x_{\varphi_2}) < a^0(x_{\varphi_2}) < a^0(x_{\varphi_1}).$$

is satisfied.

We go over to the direct study of the problem (III) which includes the cases $(A_1) - (A_3)$. Let us consider the boundary value problem (III) with

$$F(x,\varphi) : (0,1] \times (0,\infty) \to (0,\infty). \tag{B$_1$}$$

In condition (B_1) for $F(x,\varphi)$ we dropped index a_i, considering a general case of nonlinear dependence F of x.

We shall assume that F is a Caratheodory function, i.e.,

$$F(\cdot, s) \;\; \text{measurable for all} \;\; s \in R, \tag{B$_2$}$$

$$F(x,\cdot) \;\; \text{is continuous a.e. for} \;\; x \in]0,1], \tag{B$_3$}$$

and the following conditions hold

$$\int_0^1 s(1-s)Fds < \infty. \tag{B$_4$}$$

$$\partial F/\partial \varphi > 0, \;\; \text{i.e.,} \;\; F \text{ is increasing in } \varphi. \tag{B$_5$}$$

There are $\gamma(x) \in L^1(]0,1])$ and $\alpha \in R$, $0 < \alpha < 1$ such that

$$|F(x,s)| \le \gamma(x)(1 + |s|^{-\alpha}), \;\; \forall (x,s) \in]0,1] \times R. \tag{B$_6$}$$

We are intersted in a positive classical solution of equation (III), i.e., $\varphi > 0$ in P for $x \in]0,1]$ and $\varphi \in C([0,1]) \bigcap C^2(]0,1])$. The problem (III) is singular, therefore, condition (B_1) is not fulfilled on the interval $\varphi \in (0,\infty)$ and in this connection, the well-known theorems (see Amann [1]) on existence of lower and upper solution in cone P does not work. It follows from Theorem 3.1, since F in (III) is increasing in φ, then $\varphi < w$ for $x \in]0,1]$, where φ and w satisfy the differential inequality (3.3).

Theorem 3.2: *Assume conditions $(B_2) - (B_6)$. Then there exists a positive solution $\varphi \in C([0,1]) \bigcap C^2(]0,1])$ of the boundary value problem* (III).

Proof: Let $\varphi > 0$ is a solution of problem (III). By Theorem 3.1 $\varphi < w$ for $x \in]0,1]$. Take $\epsilon > 0$ and consider equation

$$\varphi_\epsilon^{''} = j_x \frac{1 + \varphi_\epsilon + \epsilon}{\sqrt{(1+\varphi_\epsilon+\epsilon)^2 - 1 - (x_{a_i})^2}} \triangleq F_\epsilon(j_x, \varphi_\epsilon + \epsilon, x_{a_i}). \tag{3.11}$$

$$\varphi_\epsilon(0) = 0, \quad \varphi_\epsilon(1) = \varphi_L.$$

Let w and φ are upper and lower solutions of equation (3.11) (below, in Proposition 3.1 is shown that such solutions really exist). Hence the theorem on monotone iterations (see Heikkila [1]) gives an existence of classical solution φ_ϵ of equation (3.11), which satisfies $w > \varphi_\epsilon > \varphi$ for $x \in]0,1]$ and is bounded in C. Thus $F_\epsilon(j_x, \varphi_\epsilon + \epsilon, x_{a_i})$ is bounded and there exists uniform limit $\lim_{\epsilon\to 0} \varphi_\epsilon = \varphi$. It follows from the last, if $0 < \eta < \frac{1}{2}$, then $\lim_{\epsilon\to 0} F_\epsilon(j_x, \varphi_\epsilon + \epsilon, x_{a_i}) = F(j_x, \varphi, x_{a_i})$ uniformly on $[\eta, 1-\eta]$ and $\varphi > 0$ for $x \in [\eta, 1-\eta]$.

Since φ_ϵ is uniformly converged on $[0,1]$, then it implies existence $\lim_{\epsilon\to 0}\varphi_\epsilon^{'}(\eta)$. Therefore there exists $\lim_{\epsilon\to 0}\varphi_\epsilon^{''}(x)$ on the compact subspaces (0,1) and $\{\varphi_\epsilon^{'}\}$ is uniformly converged on (0,1) to a differentiable function $\varphi^{'}$ on $[\eta, 1-\eta]$. From the last it follows that φ is twice differentiable on $[\eta, 1-\eta]$, $\varphi^{''} = F(j_x, \varphi, x_{a_i})$, $x \in [\eta, 1-\eta]$ and $u \in C([0,1]) \bigcap C^2(]0,1])$ is a positive solution of the problem (III).

The Theorem 3.2 is proved.

Remark 3.3: Delicate moment in the proof of Theorem 3.2 is connected with finding of a lower φ and an upper w solutions for perturbed problem (3.11). As a lower solution we can take solution of equation (A_1) (semitrivial solution φ), then an upper solution will be, for example, maximal solution of equation (A_1).

Application of monotone iteration techniques to the equation (III) gives an existence of maximal solution $\bar{\varphi}(x, j_x)$ such that

$$\varphi(x, x_j) \le \bar{\varphi}(x, x_j) < w(x) \;\text{ for }\; x \in]0,1]. \tag{3.12}$$

Proposition 3.1: *Let* $0 < c \le j_x \le j_x^{max}$. *Then equation* (A_1)

$$\varphi^{''} = F(j_x, \varphi, 0) = j_x \frac{1+\varphi}{\sqrt{\varphi(2+\varphi)}},$$

$$\varphi(0) = 0, \;\; \varphi(1) = \varphi_L$$

has a lower positive solution

$$u_0 = \delta^2 x^{4/3}, \tag{3.13}$$

if

$$4\delta^3 \ge 9 j_x^{max}(1+\delta^2)/\sqrt{2+\delta^2} \tag{3.14}$$

and an upper positive solution

$$u^0 = \alpha + \beta x \;\; (\alpha, \beta > 0) \tag{3.15}$$

with

$$\varphi_L \ge \delta^2, \tag{3.16}$$

where δ *is defined from* (3.14).

Remark 3.4:. Square root is taking as $\sqrt{|\varphi(2+\varphi)|}$ in the case of negative solutions. Here $u^0 = -\epsilon x$ is an upper solution, and $u_0 = -2+\epsilon$ is a lower solution $(0 < \epsilon < 1)$. Hence equation (A_1) has the negative solution only for $0 < \varphi_L < -2$ because $F(x, -2) = -\infty$.

It follows from (3.14), (3.16) that a value of current is limited by the value of electrostatic potential on the anode φ_L

$$j_x \le j_x^{max} \le \mathcal{F}(\varphi_L). \tag{3.17}$$

Analysis of lower and upper solutions (3.13), (3.15) exhibits that for $\delta^2 = \varphi_L > 2$ and $\alpha = \beta \le 1$ interval in x between lower and upper solutions is decreased, and for the large values of the potential φ_L diode makes on regime $\varphi_L x^{4/3}$.

Proposition 3.2: *Let* $0 < c \le j_x \le j_x^{max}$. *Then equation* (A_4)

$$a^{''} = G(j_x, \varphi^0, a) = j_x \frac{a}{\sqrt{(1+\varphi^0)^2 - 1 - a^2}}, \;\; a(0) = 0, \;\; a(1) = a_L$$

with a lower solution $a_0 = 0$ and an upper solution $a^0 = u^0 > 0$, conditions (3.14), (3.16) has an unique solution $a(x, j_x, c)$, which is positive, moreover

$$0 \le a_L \le \sqrt{\varphi^0(2+\varphi)}. \tag{3.18}$$

Proof: The positive solution of problem (A_4) is concave and be found as a solution of initial problem with $a(0) = 0$, $a'(0) = c$, where c is a shooting parameter. The solution $a = a(x, j_x, c)$ is unique and strongly decreasing in c because the right part of differential equation is decreasing in a. The least nonnegative solution is $f(x, j_x, 0) = 0$ and for $0 \le a_L \le \sqrt{\varphi^0_L(2+\varphi^0_L)}$ there exists only one solution and no positive solutions for other values a_L.

Remark 3.5: The problem (A_5) is considered by analogy with problem (A_4), change of an upper solution $a^0 = u^0$ to a lower $a^0 = u_0$ one and $0 \le a_L \le \sqrt{\varphi_{0L}(2+\varphi_{0L})}$.

Following to the definition 3.2 and Propositions 3.1, 3.2, solutions of the problems (III), (IV) we can write in the form (Fig. 3):

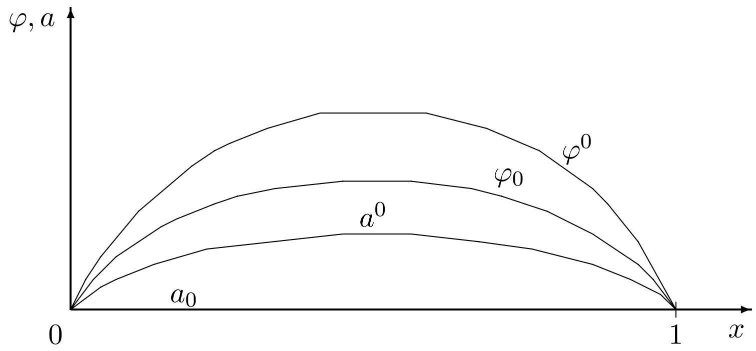

Figure 3: location of lower (φ_0, a_0) and upper (φ^0, a^0) solutions

lower-lower $(\varphi_0, a_0))$:

$$\varphi_0 = u_0 = \delta^2 x^{4/3}, \;\; a_0 = 0, \;\; \varphi_L \ge \delta^2;$$

upper-lower (φ^0, a_0):

$$\varphi^0 = u^0 = \alpha + \beta x, \;\; a_0 = 0, \;\; \delta^2 \le \varphi_L \le \mathcal{C}, \;\; \mathcal{C} = max\{\alpha, \beta\};$$

lower-upper (φ_0, a^0):

$$\varphi_0 = u_0 = \delta^2 x^{4/3}, \;\; a^0 = u^0, \;\; \varphi_L \ge \delta^2, \;\; a_L \le \sqrt{(u_0(2+u_0)};$$

upper-upper (φ^0, a^0):

$$\varphi^0 = u^0 = \alpha + \beta x, \;\; a^0 = u^0, \;\; \varphi_L \le \mathcal{C}, a_L \le a^0 \le u^0.$$

4. Existence of solutions of system (I).

In the previous section we demonstrated the existence of semitrivial solutions of system (I). Here we show the existence of solutions for the complete system (I) using the following McKenna-Walter theorem.

Theorem 4.1: (see McKenna, Walter [5]) *Assume conditions* $(B_1)-(B_6)$. *We assume that there exists the ordered pair* $(\underline{u},\bar{u})$ $-$ *lower and upper solutions, i.e.,*

$$\underline{u},\bar{u}\in C_{loc}((0,1])^2\bigcap C([0,1])^2,\ \ \underline{u}\leq\bar{u}\quad]0,1]$$

$$\underline{u}(0)\leq 0\leq\bar{u}(0),\ \ \underline{u}(1)\leq u_L\leq\bar{u}(1);\ \ u_L\overset{\triangle}{=}(\varphi_L,a_L),$$

$$\forall x\in]0,1]:\ \forall z\in R^2,$$

$$\underline{u}(x)\leq z\leq\bar{u}(x),\ \ z_k=\underline{u}_k(x);$$

$$-\underline{u}_k''(x)\geq h_k(x,z)\tag{4.1}$$

and

$$\forall x\in]0,1]:\ \forall z\in R^2,$$

$$\underline{u}(x)\leq z\leq\bar{u}(x),\ \ z_k=\bar{u}_k(x):$$

$$-\bar{u}_k''(x)\leq h_k(x,z)\tag{4.2}$$

for all $k\in\{1,2\}$. *Then there exists a solution* $u\in C^2((0,1])^2\bigcap C([0,1])^2$ *of the problem*

$$-u''=h(\cdot,u(\cdot))\quad]0,1]$$

$$u(0)=0,\ \ u(1)=u_L.$$

For keeping of ordering of lower and upper solutions in Theorem 4.1 (in cone P) we write differential inequalities (4.1), (4.2) in the following form

$$\forall z\in[v(x),w(x)],\ \ z_1=w_1(x):$$

$${}^{\pm}w_1''(x)\overset{(\geq)}{\leq}{}^{\pm}F_1(w_1(x),z_2)$$

$$\forall z\in[v(x),w(x)],\ \ z_1=v_1(x):$$

$${}^{\pm}v_1''(x)\overset{(\leq)}{\geq}{}^{\pm}F_1(v_1(x),z_2)$$

$$\forall z\in[v(x),w(x)];\ \ z_2=w_2(x)$$

$${}^{\pm}w_2''(x)\overset{(\geq)}{\leq}{}^{\pm}F_2(z_1,w_2)$$

$$\forall z\in[v(x),w(x)];\ \ z_2=v_2(x)$$

$${}^{\pm}v_2''(x)\overset{(\leq)}{\geq}{}^{\pm}F_2(z_1,v_2).$$

Remark 4.1: Change of signs with $(+)$ to $(-)$ in differential inequalities is connected with adjustment of signs and ordering $(\leq)$ of lower (upper) solutions of system (I) in Definition 2.2 and lower (upper) solutions in Theorem 4.1.

From the last relations we obtain

$$\begin{cases} w''(x) = F_1(w_1(x), 0) \le F_1(w_1, z_2) \\ v_1''(x) \ge \sup_{z_2} F_1(v_1(x), z_2) \end{cases}, \tag{4.3}$$

$$\begin{cases} w_2''(x) \le F_2(z_1, w_2) \\ v_2''(x) \ge \sup_{z_1} F_2(z_1, v_2) \end{cases}. \tag{4.4}$$

From inequality $v_2''(x) \ge \sup_{z_1} F_2(z_1, v_2)$, we get estimations to the value of magnetic field on the anode a_L

$$a_L \le \frac{j_x}{2} \le \frac{j_x^{max}}{2} \le \frac{\mathcal{F}(\varphi_L)}{2} \tag{4.5}$$

taking account of (3.17) and $\Theta_L > 0$. Under realization of estimation (4.5) the diode works in noninsulated regime, moreover, the value a_L is limited by value of electrostatic potential on the anode φ_L with a critical value $\varphi_L = 2$. In increasing of magnetic potential a_L the diode transfers in isolated regime that leads to more complicated problem with free boundary.

Thus we have the following main result of this paper.

Theorem 4.2: *Assume conditions (B_2), (B_3), (B_6) and inequalities (3.14), (3.17), (4.5). Then the problem (I) possesses a positive solution in cone P such that*

$$\begin{cases} \varphi_0'' \ge j_x F(\varphi_0, z_2), \quad z_2 \in [0, \varphi^0] \\ (\varphi^0)'' \le j_x F(\varphi^0, z_2), \quad z_2 \in [0, \varphi^0] \end{cases},$$

$$\begin{cases} a_0'' \ge G(j_x, z_1, a_0), \quad z_1 \in [\varphi_0, \varphi^0] \\ (a^0)'' \le G(j_x, z_1, a^0), \quad z_1 \in [\varphi_0, \varphi^0] \end{cases},$$

where $\varphi_0 = \delta^2 x^{4/3}$ is a lower solution of problem (A_1), $\varphi^0 = \alpha + \beta x$ $(\alpha, \beta > 0)$ is an upper solution of problem (A_1) with condition $\varphi_L \ge \delta^2$; $a_0 = 0$ is a lower solution of problem (A_4) with condition $0 \le a_L \le \sqrt{\varphi^0(2 + \varphi^0)}$.

Theorem 4.2 may be used to the construction of the minimal and maximal solution of (I) on the basis of monotone-iteration method in Heikkila [8].

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