# Interlaced Branching Equations and Invariance in the Theory of Nonlinear Equations

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The solutions of nonlinear equations can be depended of one or some free parameters. In this paper it is assumed that a kernel of linearized operator is nontrivial [1]. Then the presence of free parameters is connected with properties of range of mapping. If in addition the problem is group-invariant [10], then all or part of free parameters have a group sense [2-6, 10, 11]. In the branching theory it is necessary to know the domain of free parameters both for qualitative and asymptotic analysis [1-5], development of iterative methods [6-8] and in applications [11, 15, 16].

The concept of interlaced equation permits to simplify calculations and consider a various classes of branching solutions with unique point of view. The main results in this direction, under the condition of group- invariance, were developed by eastern contributors [2-5] and western ones [19]. There is extensive bibliography in [5, 19].

In this paper we analyze the appearance of free parameters in branching solutions of the general nonlinear equations in Banach spaces. We consider the sufficient conditions to reduction of number of equations in branching system [1]. New methods for simplification of branching equations extending the possibility of effective algorithmizaion of bifurcation analysis are given. The results of P.2 of this work are obtained collaborate with a graduate student V.R.Abdullin.

**P.1.** Let $E_1$, $E_2$ are Banach spaces, $\Lambda$ be normalized space. We consider the equation

$$F(x,\lambda) \stackrel{def}{=} Bx - R(x,\lambda) = 0 \tag{1}$$

where $B : D \subset E_1 \to E_2$ be closed Fredholm operator with dense domain of definition in $E_1$, $\lambda \in \Lambda$, $\{\varphi_1, \dots, \varphi_n\}$ is a basis in $N(B)$, $\{\psi_1, \dots, \psi_n\}$ is a basis in $N(B^*)$; systems $\{\gamma_1, \dots, \gamma_n\}$, $\{z_1, \dots, z_n\}$ are biorthogonal to these basis. The operator $R(x, \lambda)$ with values to $E_2$ is defined, continuous and continuously differentiable by Frechet with respect to $x$ in a neighbourhood of zero; $R(0, 0) = 0$, $R_x(0, 0) = 0$. It is required to construct the solutions $x \to 0$ at $\lambda \to 0$. We shall seek solutions in the form:

$$x = (\xi, \varphi) + \Gamma y \tag{2}$$

where $y$ is an unique small solution of equation

$$y = R((\xi, \varphi) + \Gamma y, \lambda), \tag{3}$$

$$(\xi, \varphi) = \sum_1^n \xi_i \varphi_i,$$

$\Gamma = (B + \sum_1^n < \cdot, \gamma_i > z_i)^{-1}$ is a bounded operator [1]. The parameter $\xi \in R^n$ satisfies the branching equation

$$< y((\xi, \varphi), \lambda), \psi_i > \stackrel{def}{=} L_i(\xi, \lambda) = 0, \quad i = 1, \dots, n. \tag{I}$$

Let us introduce the linear operators $S \in Z(E_1 \to E_1)$ and $K \in Z(E_2 \to E_2)$ interlaced by operators $B$ and $R(x, \lambda)$:

$k)$
$$BS = KB,$$

$$R(Sx, \lambda) = KR(x, \lambda) \ \text{ for } \ \forall x, \lambda \in \Omega.$$

The operators $S$, $K$ can be projectors as in [14] or if the problem $G$ be invariant, they can be a parametric reprezentations of $G$-group. We'll call the equation (1) $(S, K)$- interlaced, if the condition $k)$ holds.

We find the form in which the interlacing property $k)$ is inherited by branching equation (I). Introduce the notations

$$E_1^n = span\{\varphi_1, \dots, \varphi_n\}, \quad E_{2n} = span\{z_1, \dots, z_n\},$$

$$E_1^{n*} = span\{\gamma_1, \dots, \gamma_n\}, \quad E_{2n}^* = span\{\psi_1, \dots, \psi_n\}.$$

Let

$(i)$
$$S\varphi = T'\varphi, \quad Kz = D'z, \quad K^*\psi = M\psi, \quad S^*\gamma = C\gamma.$$

Here $T$, $D$, $M$, $C$ are matrices $[n \times n]$.

We'll search the solutions as:

$$x = (\xi, S\varphi) + \Gamma y_s \tag{4}$$

where $y_s = y((\xi, S\varphi), \lambda)$ is the unique small solution of equation

$$y = R((\xi, S\varphi) + \Gamma y, \lambda). \tag{5}$$

Parameter $\xi$ satisfies the branching equation

$$< y((\xi, S\varphi), \lambda), \psi_i) \stackrel{def}{=} L_i(T\xi, \lambda) = 0, \;\; i = 1, \ldots, n \qquad (II)$$

by condition $i$).

**Theorem 1.** *Assume $k$), $i$), moreover $C = D$. Then*

$$L(T\xi, \lambda) = ML(\xi, \lambda), \tag{6}$$

*i.e. equation (I) is $(T, M)$- interlaced.*

Proof. Equations (3), (5) possess unique small solutions

$$y = y((\xi, \varphi), \lambda) \tag{7}$$

$$y_s = y((T\xi, \varphi), \lambda) \tag{8}$$

by conditions $k$), $i$). Substituting (7) into (3) we obtain identity. Having operated by $K$ on this identity taking account of $k$), $i$) and identity $S\Gamma = \Gamma K$ we obtain:

$$Ky((\xi, \varphi), \lambda) = R((T\xi, \varphi), \lambda) + \Gamma K(y(\xi, \varphi), \lambda), \lambda).$$

Because of uniqueness of small solutions of equation (5)

$$y((T\xi, \varphi), \lambda) = Ky((\xi, \varphi), \lambda). \tag{9}$$

By projecting identity (9) to $E_{2n}$ we obtain the required equality (6).

Let us introduce instead of $k$) the following condition

$k'$) $\qquad BS = KB, \;\; KR(Sx, \lambda) = R(Sx, \lambda) \;$ for $\; \forall x, \lambda \in \Omega.$

Therefore we have the following result:

**Theorem 2.** *Assume* $k'$), $i$), $C = D$. *Then vector* $L(T\xi, \lambda)$ *is fixed one of matrix* $\mu$ *for* $\forall \xi, \lambda$.

Proof applies the method of susuccessive approximations.

**Remark.** In contrast to Loginov and Trenogin Theorem (1975) on inheritance of the group symmetry by the branching equation, we do not suppose the parametric continuity of operators $S$, $K$ in the Theorems 1, 2.

**Corollary 1.** Let the branching equation (I) is $(T, M)$-interlaced. Let $rankM = q$, $\{e_i^*\}_{i=1}^r$ is a basis in $N(M^*)$, $r = n - q$,

$$e^* \begin{pmatrix} 1, & \ldots, & r \\ k_1, & \ldots, & k_r \end{pmatrix} - \text{ is a rank minor of matrix } \parallel e_{ij}^* \parallel_{i=1,\ldots,r;\ j=1,\ldots,n} .$$

If here $(\xi, \lambda)$ satisfies to $q$ equations

$$L_i(T\xi, \lambda) = 0, \ \ i = (1, \ldots, n) \backslash (k_1, \ldots, k_r),$$

then $(\xi, \lambda)$ satisfies the remaining equations of system (II).

Proof follows from the Theorem 1 and identity

$$\sum_{s=1}^{n} e_{is}^* L_s(T\xi, \lambda) = 0 \ \text{ in which } \ \det \parallel e_{is}^* \parallel_{i=1,\ldots,r,\ s=k_1,\ldots,k_n} \neq 0.$$

**P.2.** In this section $(T(\alpha), M(\alpha))$ be parametric matrices and it is assumed that

$$f) \qquad L(T(\alpha)\xi, \lambda) = M(\alpha) L(\xi, \lambda)$$

for $\forall \xi, \lambda$ from the neighborhood of zero and $\forall \alpha \in G$, where $G$ is a domain of Euclidean space, $0 \in G$, $T(0) = E$, $\det M(\alpha) \mid_{\forall \alpha \in G} \neq 0$.

Let $c = (c_1, \ldots, c_n) \in R^n$, $R^n = R_q^n \oplus R_{n-q}^n$, where

$$R_q^n = \{c \in R^n : c_i = 0,\ i = n_{q+1}, \ldots, n_n\}.$$

$$R_{n-q}^n = \{c \in R^n : c_i = 0,\ i = n_1, \ldots, n_q\},$$

$\{n_1, \ldots, n_n\}$ is a permutation of numbers $\{1, \ldots, n\}$.

**Definition 1.** If $\exists\alpha_c \in G$ such that $M(\alpha_c)c \in R^n_q$, then we shall say, the trajectory $0(c)$ pass through subspace $R^n_q$. If $c \in R^n_{n-q}$ and $\exists\alpha_c \in G$ such that $M(\alpha_c)c \notin R^n_{n-q}$, then we shall say, the trajectory $0(c)$ leaves $R^n_{n-q}$.

Let us introduce the matrix $M_0(\alpha)$ corresponding to minor

$$M\begin{pmatrix} n_1, & \ldots, & n_q \\ & & \\ n_{q+1}, & \ldots, & n_n \end{pmatrix}.$$

By method by contradiction we proof:

**Property 1.** Let

1) $q \geq n/2$ and $\exists\alpha_0 \in G$ such that $rank M_0(\alpha_0) = n - q$ or

2) $q \leq n - 2$ and at least one row of matrix $M(\alpha)$ contains of linear independent functions.

Then the trajectory $0(c)$ corresponding to matrix $M(\alpha)$ leaves $R^n_{n-q}$ for any nonzero vector $c$ from $R^n_{n-q}$.

Using identity (6) and definition 1 we obtain the following theorem:

**Theorem 3.** *Assume $f$) and a couple $(\xi^*, \lambda^*)$ satisfies to $q$ equations of branching system (II) for $\forall\alpha \in G$, i.e.*

$$L_i(T(\alpha)\xi, \lambda) = 0, \quad i = (n_1, \ldots, n_q). \tag{11}$$

*Let one of the following conditions holds:*

*1) trajectory of any nonzero vector $c$ from $R^n$ pass through $R^n_q$;*

*2) trajectory of any nonzero vector $c$ from subspace $R^n_{n-q}$ leaves one.*

*Then a couple $(T(\alpha), \xi^*, \lambda^*)$ satisfies the complete branching system (I) for $\forall\alpha \in G$.*

**Remarks.**

a) Theorem 3 reinforces the similar results from [3]. In fact, if $S(\alpha)$-group of linear operators acting to $E^n_1$ is $q$- stationary (see definition 1 in [3]), then condition 1) of the Theorem 3 holds and we obtain the results of Theorem 1 from [3].

b) If $M(\alpha) = T(\alpha)$, then in the Theorem 3 we can change conditions 1), 2) to equivalent ones:

$1'$) $\forall \varphi \in E_1^n \;\; \exists \alpha_\varphi \in G$ such that $P_2 S(\alpha_\varphi)\varphi = 0$;

$2'$) $\forall \varphi \in E_1^n \;\; \exists \alpha_\varphi \in G$ such that $P_1 S(\alpha_\varphi) P_2 \varphi = 0$

where

$$P_1 = \sum_{i=1}^{q} < \cdot, \gamma_{n_i} > \varphi_{n_i},$$

$$P_2 = \sum_{i=q+1}^{n} < \cdot, \gamma_{n_i} > \varphi_{n_i}.$$

**P.3. Some problems and applications.**

In the case of interlaced branching equations of potential type [12, 13, 17] it will be interesting to investigate the phenomenon of domain stratification of free parameters onto the separate hypersurfaces.

In applications this fact corresponds to decomposition of space of coefficients of projection $P_{N(B)}x$ onto the direct sum of subspaces with introduction of coordinate system in every subspace. The rational choice of coordinate system yields reduction to the number of equations in branching system. We can show how this choice depends from invariants of potential of branching system.

Moreover, in some cases all desired solutions of equation (1) read as:

$$x = (t(\alpha)\xi, \mu) + \Gamma y \tag{12}$$

where $t(\alpha)$ is matrix $[n \times q]$, $1 \leq q \leq n-1$, $\mu \in R^q$. If we take matrix $t(\alpha)$ as a function of $q$-invariants of potential of branching equation (I), then we define the parameter $\mu$ from $q$- branching equations independent of parameter $\alpha$.

In series important applications (see, for example, [15], [16]) this fact jields not only the chance to proof the existence theorems of branching solutions of nonlinear boundary-value problems, but to construct all such solutions in the form (12), using degree geometry [18] and our method of successive approximations from [6-8].

## References


[1] Vainberg,M.M., Trenogin, V.A. *Branching Theory of Solutions of Nonlinear Equations.* Walters-Noordhoff, Gronningen, 1974.

[2] Judovich,V.I. Free convection and a branching. *J.Appl.math.mech.* 31(1967), 101-111.

[3] Loginov,B.V., Trenogin,V.A. On using of group properties for the determination of multiparameter families of solutions of nonlinear equations. *Math. USSR Sbornik*, 14(1971), 3, 438-452.

[4] Loginov,B.V., Trenogin,V.A. On application of continuous groups in branching theory. *Dif.Equations*, 11(1975), 1518-1521.

[5] Loginov,B.V. *Branching Theory of Solutions of Nonlinear Equations under Group Invariance Conditions.* "FAN", Tashkent, 1985.

[6] Loginov,B.V., Sidorov,N.A. Group symmetry of the Lyapunov-Schmidt branching equation and iterative methods.*Math. USSR Sbornik*, 73(1992), 1, 67-77.

[7] Sidorov,N.A. Explicit and implicit parametrizations in the construction of branching solutions by iterative methods. *Sbornik: Mathematics* 182: 2(1995), 297-310.

[8] Sidorov,N.A. N-step iterative method in bifurcation theory. *Siberian Math.J*, 38(1997), 2, 383-395.

[9] Sidorov,N.A. $(T, \mu)$- commutativity, rank and iterations in bifurcation theory. Proceedings of 11-th Baikal International School-Seminar. Irkutsk, 5-12 Julay(1998),V.4, 169-173.

[10] Ovsyannikov,L.V. *Group Analysis of Differential Equations.* Academic Press, 1982.

[11] Loginov,B.V. Determination of the branching equation by its group symmetry-Andronov-Hopf bifurcation. *Nonlinear Analysis. Theory, Methods and Applications*, 28(1997), 12, 2033-2047.

[12] Sidorov,N.A. Variational methods in branching theory. Differential and Integral Equations, Irkutsk State University, 2(1973), 255-272.

[13] Sidorov,N.A., Trenogin,V.A. Conditions of potentiality of branching equation and bifurcation points. *Uzbek.Math.J.* 2(1992), 40-49.

[14] Sidorov,N.A., Romanova,O.A. Difference-differential equations with the Fredholm operator in the main part. Katholiek Universitet Leuven, report Tw 243, june 1996.

[15] Sidorov,N.A., Sinitsyn,A.V. Analysis of bifurcation points and nontrivial branches of solutions to the stationary Vlasov-Maxwell system. *Mathematical Notes*. 62(1997), 2, 223-243.

[16] Sidorov,N.A., Sinitsyn,A.V. On bifurcation points of the stationary Vlasov-Maxwell system with bifurcation direction. Proceedings of ECMI-98, Goteborg(1988, to appear.

[17] Trenogin,V.A., Sidorov,N.A., Loginov,B.V. Bifurcation, potentiality, group-theoretical and iterative methods. *Zeitchrift fur Angewandte mathematik und mechanik*, 76(1996), 2, 245-250.

[18] Bruno,A.D. *Power geometry in algebraic and differential equations.* Moscow: Nauka, 1998.

[19] Sattinger,D.H. Group theoretic methods in bifurcation theory. Lecture Notes in Math., 1979, 762.