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GENERALIZED SOLUTIONS OF NONLINEAR INTEGRAL-FUNCTIONAL EQUATIONS

The method of construction of the generalized solutions with a point carrier in singular part is proposed for nonlinear Volterra integral-functional equations

$$\int_0^t K(t,s)(x(s) + ax(\alpha s) + g(s^l x(s), s))ds = f(t)$$

with sufficiently smooth kernel and function f; α and a are constants, and $0 < |\alpha| < 1$. The solution is constructed as a sum of singular and regular components. The special system of linear algebraic equations is used for construction of the singular component. The regular part is constructed by method of successive approximations combined with method of undetermined coefficients. The theorems of existence and uniqueness of the generalized solutions are proved.

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1. Introduction.

We consider the nonlinear integral equation

$$\int_0^t K(t,s)(x(s) + ax(\alpha s) + g(s^l x(s), s))ds = f(t), \tag{1}$$

where kernel K, g and f are analytical functions in neighborhood of zero and

$$K(t,s) = \sum_{i=0}^{n} K^n_{n-i} t^{n-i} s^i + O((|t| + |s|)^{n+1}),\ K^n_{n-i} \neq 0, i = 0, 1, \ldots, n,\ n < l.$$

We construct the generalized solution with point support (impulse, or Dirac functions) [1] singular component as follows

$$x(t) = c_0\delta(t) + c_1\delta^{(1)}(t) + \ldots + c_n\delta^{(n)}(t) + u(t), \tag{2}$$

where $\delta(t)$ is Dirac function, $u(t)$ is regular function. The generalized solutions of linear Volterra integral equations of the first kind were studied in papers [2]-[6]. This paper generalizes the basic results discussed in papers [2]-[6]. The reader may see the monographs [8] - [10] for the state-of-the-art theory of the Volterra integral equations. We presume that theory of Volterra integral-functional equations has not been addressed yet. In first and second sections we consider the equation (1) without functional change of argument (i.e. $a = 0$). The general case $a \neq 0$ will be addressed in section three. Problems presented in first and second sections have been studied by N. A. Sidorov and D. N. Sidorov. Results presented in section three has been studied by N. A. Sidorov and A. V. Trufanov.

2. Singular component determination in case $a = 0$.

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Let space of all infinitely differentiable finite functions with support in the neighborhood $(-\rho, \rho)$ be denoted by $D_{(-\rho,\rho)}$. Let space of linear continuous functionals determined on $D_{(-\rho,\rho)}$ be denoted by $D'_{(-\rho,\rho)}$. Let the subspace of its elements (2) with nth singularity order and with support in zero be $D'_{n(-\rho,\rho)}$. The solution (2) we construct in the class $D'_{n(-\rho,\rho)}$ and this solution is supposed to satisfy the equation (1) in terms of the Sobolev-Schwartz distributions [1]. It is to be noted that for $n < l$ $\forall x \in D'_{n(-\rho,\rho)}$ the multiplication $t^l x(t) = t^l u(t)$ is a regular function. This condition allows us to solve the problem of nonlinear operations with generalized functions for equations (1) for $l > n$.

In space D' when $i, j = 0, 1, \ldots, n$, $k \geq n$ the following statements are correct

$$t^{k-i}\Theta * s^i \delta^{(j)}(s) = \begin{cases} 0 \text{ when } i \neq j \\ (-1)^j j! t^{k-j} \text{ for } i = j, \end{cases}$$

where Θ is the Heaviside function. Then

$$\int_0^t \sum_{k=n}^{\infty} \sum_{i=0}^{k} K_{k-i}^k t^{k-i} s^i (c_0\delta(s) + \ldots + c_n \delta^{(n)}(s)) ds = \sum_{j=0}^{n} (-1)^j j! \sum_{k=n}^{\infty} K_{k-j}^k t^{k-j} c_j.$$

Let us note that

$$\sum_{j=0}^{n} (-1)^j \frac{\partial^j K(t,s)}{\partial s^j}\Bigg|_{s=0} = \sum_{j=0}^{n} (-1)^j j! \sum_{k=n}^{\infty} K_{k-j}^k t^{k-j}.$$

Hence the element $x \in D'_{n(-\rho,\rho)}$ can be the solution of the equation (1) if the regular component in (2) satisfies the equation

$$\int_0^t K(t,s)(u(s) + g(s^l u(s), s)) ds = r(t, c_0, \ldots, c_n), \tag{3}$$

where

$$r(t, c_0, \ldots, c_n) = f(t) - \sum_{j=0}^{n} (-1)^j \frac{\partial^j K(t,0)}{\partial s^j} c_j. \tag{$*$}$$

Now we have to determine the parameters $c_0, \ldots, c_n$ from the system of linear algebraic equations

$$r_t^{(i)}(0, c_0, \ldots, c_n) = 0, \ i = 0, \ldots, n \tag{4}$$

with subdiagonal matrix with components $K_0^n, K_1^n, \ldots, K_n^n$ placed on the diagonal.

If these numbers are not zeros then the parameters $c_n, .., c_0$ can be consistently and uniquely determined. If some of the diagonal elements are zeros and vector $\{f(0), f'(0), \ldots, f^{(n)}(0)\}'$ meets the solvability conditions then part of parameters can remain arbitrary in the right hand side of the equation (3).

Remark. If $f^{(i)}(0) = 0$, $K_i^n = 0, i = 0, 1, \ldots, k-1$, $K_i^n \neq 0$ for $i = k, .., n$, then if $c_k = \ldots = c_n = 0$, it is possible to uniquely define the parameters $c_0, \ldots, c_{k-1}$ from the system (4).

3. Regular component determination in case $a = 0$.

Lets $a = 0$ in equation (1). To determine the regular function $u(t)$ we put $a = 0$ in equation (1). Then we need to solve this equation with known $c_0, \dots, c_n$ by method of successive approximations along with method of undetermined coefficients.

For sake of clarity we make the following notation

$$\Phi(u,t) := \int_0^t K(t,s)(u(s) + g(s^l u(s), s))ds - r(t,c) = 0. \tag{5}$$

We assume the homogeneous equation

$$\int_0^t \sum_{i=0}^n K_{n-i}^n t^{n-i} s^i x(s) ds = 0 \tag{6}$$

has the trivial solution only. This is the case if $\sum\limits_{i=0}^n K_{n-i}^n \frac{1}{i+j} \neq 0$ $j = 1, 2, \dots$. Then for any positive integer N there will be coefficients u_i which meet the following condition

$$|\Phi(u_0 + u_1 t + \ldots + u_N t^N, t)| = O(|t|^{n+N+1}). \tag{7}$$

Let the homogeneous equation (6) has the trivial solution only. Hence

$$\int_0^t \sum_{i=0}^n K_{n-i}^n t^{n-i} s^{i+j} ds \neq 0$$

for $j = 0, 1, 2, \dots$ and the coefficients u_i can be uniquely determined by method of undetermined coefficients. We make the notation

$$u^0(t) = u_0 + u_1 t + \ldots + u_N t^N \tag{8}$$

Now we can substitute the function

$$u(t) = u^0(t) + t^N v(t) \tag{9}$$

into the equation (5). We have to group and exclude terms with powers $t^i, i = n, n+1, \dots, n+N$, by taking into account equalities (4) and the polynomial $u^0(t)$ structure. The result should be differentiated wrt t. We use the successive approximations method to find function v from the equivalent integral equation

$$v = F(v,t). \tag{10}$$

Here

$$F(v,t) = \frac{1}{K(t,t)t^N} \Big\{ - \int_0^t K_t'(t,s)(u^0(s) +$$

$$+ s^N v(s) + g(s^l u^0(s) + s^{l+N} v(s), s))ds + r_t'(t,c) \Big\} - \frac{u^0(t)}{t^N}. \tag{11}$$

Let us suppose that

$$\sum_{i=0}^{n} K_{n-i}^{n} = a \neq 0. \tag{12}$$

For large enough N we can demonstrate that operator F satisfies the conditions of contraction mapping principle on the sphere $||x|| \leq r$ of the space $C_{[-\rho,\rho]}$. Indeed

$$|g(s^l(u^0(s) + s^N v_1(s), s) - g(s^l(u^0(s) + s^N v_2(s)), s)| \leq |s|^{l+N} C_1 |v_1 - v_2|$$

$\forall v_1, v_2$ from the sphere $S(0, r) \subset C_{[-\rho,\rho]}$. The kernel K meets the condition

$$|K'_t(t, s)| \leq C_2(|t| + |s|)^{n-1}$$

hence

$$\left| \frac{1}{t^{n+N}} \int_0^t K'_t(t, s) s^N ds \right| \leq \frac{C_2 2^{n-1}}{N+1}. \tag{13}$$

In above mentioned estimations the constant c exists and satisfies the following condition

$$|F(v_1, t) - F(v_2, t)| \leq \frac{c}{N+1} ||v_1 - v_2||.$$

We can fix $q < 1$ and choose $N > \frac{c}{q} - 1$. Then operator F in neighborhood $||v|| \leq r$ in space $C_{[-\rho,\rho]}$ will be contracting with aspect ratio q.

Due to the estimation (7) $|F(0, t)| = O(|t|)$ the following statement is correct and $\max\limits_{|t| \leq \tilde{\rho}} |F(0, t)| \leq (1 - q)r$ for $\tilde{\rho} \in (0, \rho]$. Consequently, the contraction operator F maps the neighborhood $||v|| \leq r$ in space $C_{[-\tilde{\rho},\tilde{\rho}]}$ into itself. Taking into account previous statements we can formulate the following theorem.

Theorem 1. *Let the following conditions are fulfilled*

$$l > n, a = 0,$$

$$\sum_{i=0}^{n} K_{n-i}^{n} \frac{1}{i+j} \neq 0, \; j = 1, 2, \ldots,$$

$$K_{n-i}^{n} \neq 0, \; i = 0, 1, \ldots, n,$$

$$\sum_{i=0}^{n} K_{n-i}^{n} \neq 0.$$

Then equation (1) for $a = 0$ has the unique solution (2), (9) in space of functions $D'_n(-\tilde{\rho}, \tilde{\rho})$, where constants $c_0, \ldots, c_n$ are determined from equations (4), coefficients $u_0, \ldots, u_N$ are calculated by the method of undetermined coefficients from equation (5). Continuous function $v(t)$ is built by the method of successive approximations from equation (10).

Remarks.

1. In analytical case the regular component is analytical function in the neighborhood of zero. The Taylor's coefficients of this function can be determined by the method of undetermined coefficients from equation (5).

2. Analyticity of K, g, f in theorem 1 can be replaced by sufficient smoothness of these functions.

3. If $f^{(i)}(0) = 0$, $i = 0, 1, \dots, n$, then in solution (2) all $c_i = 0$. This is corresponds to the conventional Volterra integral equation.

4. If in the conditions of theorem 1 some of the elements K_i^n, $i = 0, \dots, n$ are zeros and the system (4) is solvable, then solution (2),(9) will depend on k arbitrary parameters, where $k = n + 1 - r$, r is rank of the matrix of linear system (4). Theorem 1 can be extended.

Theorem 2. *Lets in conditions of theorem 1* $\sum\limits_{i=0}^{n} K_{n-i}^n = 0$ *and*

$$\frac{\partial^i K(t,s)}{\partial t^i}|_{s=t} = 0, \ i = 0, 1, \dots, p-1,$$

$$\frac{\partial^p K(t,s)}{\partial t^p}|_{s=t} = O(t^{n-p}), \ p \leq n.$$

Then results of theorem 1 remain correct.

To proof this theorem we have to take into account that on the basis of Taylor's formula in the conditions of Theorem 2 the kernel can be presented as follows $K(t,s) = (t-s)^p Q(t,s)$, where $|Q(t,t)| = O(|t|^{n-p})$. In order to determine function v we need to differentiate equation (3) $(p+1)-$times we should put

$$F(v,t) = \frac{1}{K_t^(p)(t,t)t^N}\Big\{ -\int\limits_0^t K_t^{(p+1)}(t,s)(u^0(s) + s^N v(s) +$$

$$+ g(s^l(u^0(s) + s^N v(s)), s) ds + r_t^{(p+1)}(t,c)\Big\} - \frac{u^0(t)}{t^N}.$$

Example:

$$\int\limits_0^t (t^2 + ts - 2s^2)(x(s) + s^5 x^2(s)) ds = 1 + t + t^2 + t^3. \tag{14}$$

Let conditions of the theorem 2 are fulfilled for $n = 2, l = 5/2, p = 1$. In class D' the solution $x(t) = \delta(t) - \delta^{(1)}(t) - \frac{1}{4}\delta^{(2)}(t) + \frac{-1+\sqrt{1+t^5 24/5}}{2t^5}$ exists. The solution's singular components is determined in conformity to equalities (4). For regular component we have the following equation

$$\int\limits_0^t (t^2 + ts - 2s^2)(u(s) + s^5 u^2(s)) ds = t^3. \tag{15}$$

Equation (15) has analytical solution

$$u(t) = \frac{-1 + \sqrt{1 + t^5 24/5}}{2t^5}.$$

Taylor's coefficients of this solution for $t = 0$ are calculated based on the remark 1 from equation (15) in terms of undetermined coefficients method. It is to be noted that apart from this solution the following function satisfies equation (15)

$$u_2(t) = \frac{-1 - \sqrt{1 + t^5 24/5}}{2t^5}.$$

The point $t = 0$ is a pole of fifth order for this equation.

4. Solution of Volterra integral-functional equation with functionally changed argument ($a \neq 0$).

Let us apply the following conditions in equation (1) $a \neq 0, \;\; 0 < |\alpha| < 1$. We can split the equation (1) into the system of two equations

$$\begin{cases} \int\limits_0^t K(t,s)w(s)ds = f(t) & (16) \\ x(t) + ax(\alpha t) + g(t^l x(t), t) = w(t). & (17) \end{cases}$$

Below we will suppose that functions k, g, f are analytical in the neighborhood of zero. It is to be noted that the results of this paragraph will remain correct in case of sufficiently smooth functions k, g, f.

LEMMA 1. *Let the kernel $K(t,s)$ meets the conditions of theorem 1 (or theorem 2), $f(t)$ is analytical function in neighborhood of zero. Then function*

$$w(t) = c_0\delta(t) + \ldots + c_n\delta^{(n)}(t) + p(t),$$

satisfies equation (16), where constants $c_0, \ldots, c_n$ are determined from the system (4), function $p(t)$ is unique regular solution of equation

$$\int\limits_0^t K(t,s)p(s)ds = f(t) - \sum_{j=0}^{n} (-1)^j \frac{\partial^j K(t,0)}{\partial s^j} c_j$$

To proof this lemma it is enough to use the proof of the theorem 1 in case $g(t^l x(t), t) \equiv 0$.

THEOREM 3. *Let the conditions of lemma 1 are fulfilled. In addition let the following inequalities are fulfilled for* $0 < |\alpha| < 1$

$$1 + \tfrac{a}{\alpha^i |\alpha|} \neq 0, \;\; i = 0, 1, 2, \ldots, n, \tag{18}$$

$$1 + a\alpha^j \neq 0, \;\; j = 0, 1, 2, \ldots \tag{19}$$

Then equation (1) has the generalized solution

$$x(t) = \frac{c_0}{1 + \frac{a}{\alpha^0 |\alpha|}}\delta(t) + \ldots + \frac{c_n}{1 + \frac{a}{\alpha^n |\alpha|}}\delta^{(n)}(t) + u(t), \tag{20}$$

where constants $c_0, \ldots, c_n$ can be uniquely determined from the system (4), regular function $u(t)$ is analytical in the neighborhood of zero.

Proof. Due to the lemma 1 the desired generalized solution satisfies the nonlinear functional equation

$$x(t) + ax(\alpha t) + g(t^l x(t), t) = c_0\delta(t) + \ldots + c_n\delta^{(n)}(t) + p(t). \tag{21}$$

The right hand side of the equation (21) has regular and singular components. We will construct the solution in following form

$$x(t) = e_0\delta(t) + \ldots + e_n\delta^{(n)}(t) + u(t). \tag{22}$$

The generalized functions $\delta^{(i)}(\alpha t), \quad i = 0, 1, \ldots, n$ appears as result of substitution (22) into (21). It is to be noted that

$$\int\limits_{-\infty}^{\infty} \delta(\alpha t)\varphi(t)dt = \begin{cases} \int\limits_{-\infty}^{\infty} \delta(t)\varphi(\frac{t}{\alpha})\frac{dt}{\alpha}, & \alpha > 0 \\ -\int\limits_{-\infty}^{\infty} \delta(t)\varphi(\frac{t}{\alpha})\frac{dt}{\alpha}, & \alpha < 0 \end{cases} = \frac{\varphi(0)}{|\alpha|}.$$

Dirac function's k-th derivative influences on φ as follows

$$\int\limits_{-\infty}^{\infty} \delta^{(k)}(\alpha t)\varphi(t)dt = \int\limits_{-\infty}^{\infty} \delta^{(k)}(t)\varphi(\tfrac{t}{\alpha})\tfrac{dt}{|\alpha|} = (-1) \int\limits_{-\infty}^{\infty} \delta^{(k-1)}(t)\varphi^{(1)}(\tfrac{t}{\alpha})\tfrac{dt}{\alpha|\alpha|} =$$

$$= (-1)^k \int\limits_{-\infty}^{\infty} \delta(t)\varphi^{(k)}(\tfrac{t}{\alpha})\tfrac{dt}{\alpha^k|\alpha|} = (-1)^k \tfrac{\varphi^{(k)}(0)}{\alpha^k|\alpha|}.$$

Consequently,

$$\delta^{(k)}(\alpha t) = \frac{\delta^{(k)}(t)}{\alpha^k|\alpha|}.$$

If $n < l \; \forall x(t) \in D'_{n(-\rho,\rho)}$ then function $t^l x(t) = t^l u(t)$ is a regular one. Therefore, the equation

$$\begin{aligned} &e_0(1 + \tfrac{a}{|\alpha|})\delta(t) + \ldots + e_0(1 + \tfrac{a}{\alpha^n|\alpha|})\delta^{(n)}(t) + u(t) + au(\alpha t) = \\ &= c_0\delta(t) + \ldots + c_n\delta^{(n)}(t) + p(t) - g(t^l u(t), t). \end{aligned} \tag{23}$$

can be obtained as result of substitution of solution (22) into the equation (21). Due to the conditions (18), the coefficients $e_i, \quad i = 0, 1, \ldots, n$ can be determined uniquely as follows

$$e_i = \frac{c_i}{1 + \frac{a}{\alpha^i|\alpha|}}, \quad i = 0, 1, \ldots, n.$$

The coefficients substitution into the equation (23) excludes all singular elements. To determine the regular component $u(t)$ we have nonlinear functional equation

$$u(t) + au(\alpha t) = p(t) - g(t^l u(t), t) \tag{24}$$

considered in papers [14],[15].

Based on the results [14] and conditions (19) the equation (24) has the following formal solution $u(t) = \frac{f(0)}{1+a} + \sum\limits_{i=1}^{\infty} u_i t^i$. The coefficients $u_i, i = 1, 2, \ldots$ can be calculated by the method of undetermined coefficients from equation (24).

The formal solution $u(t)$ is true solution. We can demonstrate this by following proof in a similar way as the theorem 1. Function $u(t)$ can be presented as following series

$$u(t) = u_0 + u_1 t + \ldots + u_N t^N + t^N v(t), \quad v(0) = 0.$$

Method of undetermined coefficients can be exploited in order to find the coefficients $u_0, u_1, ..,$ u_N. Let this approach be denoted as $u^0(t)$.

To determine function $v(t)$ the following equation should be considered

$$v(t) = -\alpha^N a v(\alpha t) + \tfrac{p(t) - u^0(t) - a u^0(\alpha t) - g(t^l(u^0(t) + t^N v(t)), t)}{t^N} \equiv \Phi(v(t), t).$$

Let us notice here that $\forall v(t)$ the following estimation is correct

$$p(t) - u^0(t) - a u^0(\alpha t) - g(t^l(u^0(t) + t^N v(t)), t) = o(t^N), \;\; t \to 0$$

due to the structure of $u^0(t)$.

We can follow the proof of the contraction of operator (11). It is easy to demonstrate that mapping $\Phi(v(t), t)$ is constructing and translates the sphere $\|v\| < r$ into itself due to the choice of N and radius of zero's neighborhood $|t| < \rho$. Hence $v(t) = \lim\limits_{k \to 0}(v_k(t))$ where $v_k(t) = \Phi(v_{k-1}(t), t), \;\; v_0(t) = 0$.

It is to be noted that all terms of the sequence $\{v_k\}$ are analytical functions in some neighborhood of zero and sequence has uniform convergence. Therefore function $v(t)$ will be analytical in some neighborhood of zero.

Let us demonstrate that theorem 3 can be extended by excluding of the conditions (19). For sake of clarity lets $\exists k : a\alpha^k + 1 = 0$. The following definition will be used below.

Definition. Continuous function $u(t)$ has logarithm-polynomial asymptotics of N-th order, if there are coefficients $u_{ik}, \;\; k = 0, 1 \ldots, i, \;\; i = 0, \ldots, N$ which satisfies the condition

$$\lim_{t \to 0} \frac{1}{t^N}(u(t) - \sum_{i=0}^{N} \sum_{k=0}^{i} u_{ik} t^i \ln^k t) = 0.$$

Function $u^0(t) = \sum\limits_{i=0}^{N} \sum\limits_{k=0}^{i} u_{ik} t^i \ln^k t$ is named as asymptotic logarithm-polynomial approach of N-th order of function $u(t)$.

Theorem 4. *Let the conditions of lemma 1 are fulfilled. Lets* $a\alpha^k + 1 = 0$ *and* $0 < |\alpha| < 1$ *in (1) Then equation (1) has solution (22) with following regular component*

$$u(t) = \sum_{i=0}^{k-1} u_{i0} t^i + t^k (u_{k1} lnt + c) + \sum_{i=k+1}^{N} t^i \sum_{j=0}^{i} u_{ij} \ln^j t + o(t^N), \tag{25}$$

where $c = u_{k0}$ *is free parameter. Coefficients* u_{ij} *of asymptotic* $u^0(t)$ *are determined by the method of undetermined coefficients in following sequence* u_{00}; $\underbrace{u_{11}, u_{10}}$; $\underbrace{u_{22}, u_{21}, u_{20}}$; ...

Proof. We can follow the theorem 3 proof here. The solutions of equation (1) can be presented as (20), where the regular part of solution $u(t)$ satisfies the equation (24). Because $a\alpha^k = -1$

then the equation (24) possibly has no analytical solutions. Therefore we can look for the solution in more wide class of functions which can be presented as (25). We consider vectors

$$\begin{aligned}
&\vec{u^i} = (u_{ii}, u_{ii-1}, \dots, u_{i0})^T,\\
&\vec{G}_i(\vec{u^0}, \vec{u^1}, \dots, \vec{u^{i-1}}) =\\
&= (-g_{ii}(u_{00}, u_{11}, u_{10}, \dots, u_{i-10}), \dots, p_i - g_{i0}(u_{00}, u_{11}, u_{10}, \dots, u_{i-10}))^T,
\end{aligned}$$

where p_i are coefficients of Teylor decomposition of function $p(t)$ and $g_{ij}(u_{00}, u_{11}, u_{10}, \dots, u_{i-10})$ are known functions.

Substitution of series (25) into the equation (24) for $\vec{u^i}$ determination gives the matrix equation

$$\mathcal{A}_i\vec{u^i} = \vec{G}_i(\vec{u^0}, \vec{u^1}, \dots, \vec{u^{i-1}}), \tag{26}$$

where

$$\mathcal{A}_i = \left\|\begin{array}{cccccc}
(1+a\alpha^i), & 0, & 0, & 0, & \dots & 0\\
a\alpha^i \ln\alpha C_i^{i-1}, & (1+a\alpha^i), & 0, & 0, & \dots & 0\\
a\alpha^i \ln^2\alpha C_i^{i-2}, & a\alpha^i \ln\alpha C_{i-1}^{i-2}, & (1+a\alpha^i), & 0, & \dots & 0\\
\vdots & & & & (1+a\alpha^i) & 0\\
a\alpha^i \ln^i\alpha C_i^0, & & \dots & & a\alpha^i\ln\alpha C_1^0, & (1+a\alpha^i)
\end{array}\right\|$$

is matrix of dimension $(i+1)\times(i+1)$.

The first equation of the sequence of systems can be presented as $(1+a)u_{00} = f(0)$. If $1+a = 0$ and $f(0) \neq 0$ then operator equation (16) has no solutions to be presented as (25). Therefore further $(1+a\alpha^k) = 0$ for $k \in N$.

For $i = \overline{0, k-1}$ the matrices $\mathcal{A}_i$ will be reversible,

$$\vec{G}_i(\vec{u^0}, \vec{u^1}, \dots, \vec{u^{i-1}}) = (0, \dots, 0, p_i - g_{i0}(u_{00}, u_{11}, u_{10}, \dots, u_{i-10}))^T$$

and systems (26) have the solutions $\vec{u^i} = (0, \dots, 0, u_{i0})^T$.

If $i = k$ then $\mathrm{R}ank\mathcal{A}_k = k$, matrix $\mathcal{A}_k$ is singular, and first k elements are equal to zero in vector $\vec{G}_k$. Hence $\vec{u^k} = (0, \dots, 0, u_{k1}, c)^T$, where c is arbitrary parameter.

If $i > k$ then matrix $\mathcal{A}_i$ is reversible and vector $\vec{G}_i$ depends on c and $\vec{u^0}, \vec{u^1}, \dots, \vec{u^{i-1}}$.

Thus functional equation (17) has logarithmic-polynomial asymptotic

$$u^0(t) = \sum_{i=0}^{k-1} u_{i0}t^i + t^k(u_{k1}\ln t + c) + \sum_{i=k+1}^{N} t^i \sum_{j=0}^{i} u_{ij}\ln^j t,$$

where c is free constant. Exactly as in case of the theorem 3 we can demonstrate that function

$$u(t) = u^0(t) + t^N v(t), \quad v(0) = 0,$$

satisfies the certain equation with construction operator for $v(t)$. This is the reason why this function can be uniquely determined.

Example: To demonstrate this technique let us consider the equation

$$\int_0^t e^{t-s}(x(s) + ax(\alpha s) + s^2x^2(s))ds = 2 + t, \tag{28}$$

a, α are constants, $0 < |\alpha| < 1$. We can represent the equation (28) as the system of equations

$$\begin{cases} \int\limits_0^t e^{t-s} w(s) ds = f(t) \\ x(t) + ax(\alpha t) + t^2 x^2(t) = w(t). \end{cases}$$

On the basis of theorem 1 we have $w(t) = 2\delta(t) + (1-t)$.
Lets consider the functional equation

$$x(t) + ax(\alpha t) + t^2 x^2(t) = 2\delta(t) + (1-t).$$

If $a\alpha^i + 1 \neq 0$ then on the basis of theorem 3 the equation (28) has the following unique solution

$$x(t) = \frac{2}{1 + a/|\alpha|}\delta(t) + \frac{1}{1+a} - \frac{t}{1+a\alpha} + o(t).$$

Let $a\alpha + 1 = 0$ then due to the theorem 4 the following one-parameter family of the generalized functions will be the solution of equation (28)

$$x(t) = \frac{2}{1 + a/|\alpha|}\delta(t) + \frac{1}{1+a} - \frac{t \ln t}{a\alpha \ln \alpha} + ct + o(t),$$

here c is free parameter.

Conclusion.

In general case the equation (1) has a few bifurcating solutions. Such solutions can be constructed based on the results of this work in combination with known methods of bifurcation theory [13].

Results of the theorems 1–4 combined with results presented in papers [4], [11] can be generalized on the systems and integral-operator equations (1), where kernel K is linear, and g is nonlinear mapping in Banach space. These results can be used in development of theory and application of differential-operator equations with Fredholm operator in main part [11], [12] in problems of nonlinear dynamics and identification [7], [8] and in other problems formulated in terms of the Volterra integral equations of the first kind.

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